A generalized Hölder type eigenvalue inequality

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\textbf{ABSTRACT}

In this note, we prove that if $A_1, \ldots, A_m$ are $n \times n$ contractive matrices and $p_1, \ldots, p_m > 0$ with $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1$, then
\begin{equation*}
\prod_{j=1}^{k} (1 - \lambda_j(|A_1 \cdots A_m|)^{p_j}) \geq \prod_{i=1}^{m} \prod_{j=1}^{k} (1 - \lambda_j(|A_i|)^{p_j})^{\frac{1}{p_i}}
\end{equation*}
for each $k = 1, 2, \ldots, n$. This generalizes an inequality given by Marcus in 1958.

\section{Introduction}

Let $\mathbb{M}_n$ be the set of $n \times n$ complex matrices. For $A \in \mathbb{M}_n$, the conjugate transpose of $A$ is denoted by $A^*$. $A \in \mathbb{M}_n$ is Hermitian if $A = A^*$. A Hermitian $A$ is positive semi-definite, $(A \geq 0)$ if all eigenvalues of $A$ are non-negative. For Hermitian matrices $A$ and $B$, write $A \geq B$ if $A - B \geq 0$. If $A \in \mathbb{M}_n$, let $\lambda_j(A)$, $j = 1, 2, \ldots, n$, be the the eigenvalues of $A$ so arranged that $|\lambda_j(A)| \geq |\lambda_{j+1}(A)|$ for $j = 1, 2, \ldots, n-1$. The singular values of a complex matrix $A$ are the eigenvalues of $|A| := (A^*A)^{1/2}$, and we denote the singular values of $A$ by $\sigma_j(A) := \lambda_j(|A|)$. The $n \times n$ identity matrix is denoted by $I_n$. $A \in \mathbb{M}_n$ is called contractive if $\sigma_1(A) \leq 1$, equivalently, $I_n \geq A^*A$. Hua [1] proved the following interesting inequality: Let $A, B \in \mathbb{M}_n$ be contractive. Then
\begin{equation*}
|\det (I_n - A^*B)|^2 \geq \det (I_n - A^*A) \det (I_n - B^*B),
\end{equation*}
which is known as Hua’s determinantal inequality in the literature (see [2, p.231]). Marcus [3] generalized the inequality (1.1) and obtained the following result.

\textbf{Theorem 1.1} [3, Theorem]: Let $A, B \in \mathbb{M}_n$ be contractive. Then
\begin{equation*}
\prod_{j=1}^{k} |\lambda_{n+1-j}(I_n - A^*B)|^2 \geq \prod_{j=1}^{k} [1 - \lambda_j(A^*A)][1 - \lambda_j(B^*B)].
\end{equation*}

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Marcus’s original proof uses the Grassmann algebra and is rather sophisticated. Very recently, Lin [4] gives an alternative proof of (1.2) by showing [4, Proposition 3.7] that for contractive \( A, B \in \mathbb{M}_n \),

\[
\sigma_j(I_n - A^*B)^2 \geq \sigma_j((I_n - A^*A)(I_n - B^*B)).
\]

(1.3)

The main result in this paper is the following.

**Theorem 1.2:** Suppose \( A_1, \ldots, A_m \in \mathbb{M}_n \) are contractive matrices, \( r \geq 1 \) and \( p_1, \ldots, p_m > 0 \) satisfy \( \frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1 \). Then for all \( 1 \leq k \leq n \), we have

\[
\prod_{j=1}^k (1 - \lambda_j(|A_1 \cdots A_m|)^{r_j}) \geq \prod_{i=1}^m \prod_{j=1}^{p_i} (1 - \lambda_j(|A_i|)^{p_i})^{\frac{1}{p_i}}.
\]

(1.4)

From this result, an extension (Corollary 2.7) of the eigenvalue inequality (1.2) of Marcus will follow.

**2. Proof of the main theorem**

First, we need some definitions on majorization [5]. For \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), let \( x[1] \geq \cdots \geq x[n] \) denote the components of \( x \) arranged in decreasing order. Suppose \( x, y \in \mathbb{R}^n \). We say that \( x \) is weakly majorized by \( y \), denoted by \( x <_w y \), if

\[
\sum_{j=1}^k x[j] \leq \sum_{j=1}^k y[j]
\]

for \( k = 1, \ldots, n \). Furthermore, if \( x <_w y \) and \( \sum_{j=1}^n x[j] \leq \sum_{j=1}^n y[j] \), then we say that \( x \) is majorized by \( y \), denoted by \( x < y \). Suppose \( x = (x_i), y = (y_i) \) with \( x_i, y_i \geq 0 \) for all \( 1 \leq i \leq n \). \( x \) is said to be log-majorized by \( y \), denoted by \( x <_{\log} y \), if

\[
\prod_{j=1}^k x[j] \leq \prod_{j=1}^k y[j] \quad \text{for all } k = 1, \ldots, n;
\]

with equality for \( k = n \). We note that if \( x < y \), then we also have

\[
\prod_{j=1}^k x[j+1-n] \geq \prod_{j=1}^k y[j+1-n] \quad \text{for all } k = 1, \ldots, n.
\]

In order to prove Theorem 1.2, we need some lemmas. The first three lemmas are standard results in matrix analysis.

**Lemma 2.1 [5, Chapter 9.E.1]:** Let \( A \in \mathbb{M}_n \). Then

\[
(\lambda_j(A)) <_{\log} (\sigma_j(A)).
\]
Lemma 2.2 [5, Chapter 9.H.1]: Let $A, B \in \mathbb{M}_n$. Then

$$
\left( \lambda_j(\|A\| B) \right) \prec \left( \frac{\lambda_j(\|A\|) \lambda_j(\|B\|)}{\log \lambda_j(\|A\|) \lambda_j(\|B\|)} \right)
$$

and

$$
\left( \lambda_j(AB) \right) \prec \left( \frac{\lambda_j(\|A\|) \lambda_j(\|B\|)}{\log \lambda_j(\|A\|) \lambda_j(\|B\|)} \right).
$$

By a result of Thompson [6], we have the following.

Lemma 2.3: Let $A \in \mathbb{M}_n$ be contractive. Then

$$
\lambda_j(I - \|A\|) \leq \lambda_j(I - \|A\|)
$$

for each $k = 1, 2, \ldots, n$.

Lemma 2.4: Let $A, B \in \mathbb{M}_n$ be contractive. Then for $r \geq 1$, we have

$$
\prod_{j=1}^{k} \left( 1 - \lambda_j(\|AB\|^r) \right) \geq \prod_{j=1}^{k} \left( 1 - \lambda_j(\|A\|^r \|B\|^r) \right)
$$

for each $k = 1, 2, \ldots, n$.

**Proof:** First suppose that $A, B \geq 0$ are contractive. For $r \geq 1$, by a result of Araki [7], we have

$$
\left( \frac{\lambda_j(AB)}{\log \lambda_j(\|AB\|^r)} \right) \prec \frac{\lambda_j(|A|^r |B|^r)}{\log \lambda_j(|A|^r |B|^r)}.
$$

Let $g(x) = -\log(1 - x)$. Then $g(x)$ is increasing and convex on $(0, 1)$. For $t > 0$, let

$$
A_t = A + tI_n, \quad \frac{1}{1 + 2t} \quad \text{and} \quad B_t = B + tI_n, \quad \frac{1}{1 + 2t}.
$$

Then $A_t, B_t$ are invertible and $\|A_t\|, \|B_t\| < 1$. Thus,

$$
0 < \lambda_n(\|A_tB_t\|) \leq \lambda_1(\|A_tB_t\|) < 1.
$$

By [5, Chapter 5, A.2.a], we have

$$
(1 - \lambda_j(\|A_tB_t\|^r)) \prec_w (1 - \lambda_j(\|A\|^r |B|^r)).
$$

Therefore,

$$
\prod_{j=1}^{k} \left( 1 - \lambda_j(\|A_tB_t\|^r) \right) \geq \prod_{j=1}^{k} \left( 1 - \lambda_j(\|A\|^r |B\|^r) \right)
$$

for each $1 \leq k \leq n$. Taking limit with $t \to 0^+$, we have $A_t \to A$ and $B_t \to B$. Hence,

$$
\prod_{j=1}^{k} \left( 1 - \lambda_j(\|AB\|^r) \right) \geq \prod_{j=1}^{k} \left( 1 - \lambda_j(\|A\|^r |B\|^r) \right).
$$

For the general case, note that $\lambda_j(\|AB\|) = \lambda_j(\|A\| |B\|^r)$. Then the result follows. \qed

**Proof of Theorem 1.2:** The case for $m = 2$ and $r = 1$ can be deduced from [8, Theorem 4.1], in which (2.6) is proven. We provide a proof here for completeness. By the matrix Young inequalities of Ando [9], given $A, B \in \mathbb{M}_n$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$, there is a
unitary matrix $U$ such that

$$U|AB^*U^*| \leq \frac{1}{p}|A|^p + \frac{1}{q}|B|^q.$$  \hspace{1cm} (2.5)$$

So, we have

$$I_n - U|A_1A_2|U^* \geq \frac{1}{p_1} (I_n - |A_1|^{p_1}) + \frac{1}{p_2} (I_n - |A_2^*|^{p_2})$$

$$\geq V \left| (I_n - |A_1|^{p_1}) \frac{1}{p_1} (I_n - |A_2^*|^{p_2}) \frac{1}{p_2} \right| V^*$$  \hspace{1cm} for some unitary $V$.

Therefore,

$$\lambda_i(I_n - |A_1A_2|) \geq \lambda_i \left( \left| (I_n - |A_1|^{p_1}) \frac{1}{p_1} (I_n - |A_2^*|^{p_2}) \frac{1}{p_2} \right| \right).$$  \hspace{1cm} (2.6)$$

So, we have

$$\prod_{j=1}^k \left( 1 - \lambda_j(|A_1A_2|) \right)$$

$$= \prod_{j=1}^k \lambda_{n+1-j} (I_n - |A_1A_2|)$$

$$\geq \prod_{j=1}^k \lambda_{n+1-j} \left( \left| (I_n - |A_1|^{p_1}) \frac{1}{p_1} (I_n - |A_2^*|^{p_2}) \frac{1}{p_2} \right| \right)$$

$$\geq \prod_{j=1}^k \lambda_{n+1-j} \left( I_n - |A_1|^{p_1} \right) \lambda_{n+1-j} \left( I_n - |A_2^*|^{p_2} \right)$$  \hspace{1cm} by Lemma 2.2

$$= \prod_{j=1}^k \left( 1 - \lambda_j(|A_1|^{p_1}) \frac{1}{p_1} (1 - \lambda_j(|A_2|^{p_2}) \frac{1}{p_2} \right).$$

Replacing $A_1$ and $A_2$ by $|A_1|^r$ and $|A_2|^r$, we have

$$\prod_{j=1}^k \left( 1 - \lambda_j \left( |A_1|^r \left| A_2^* \right|^r \right) \right) \geq \prod_{j=1}^k \left( 1 - \lambda_j \left( |A_1|^{r_1} \right) \frac{1}{p_{r_1}} (1 - \lambda_j \left( |A_2|^{r_2} \right) \frac{1}{p_{r_2}} \right).$$

Since $\lambda_j(|A_1A_2|) = \lambda_j(|A_1||A_2^*|)$, by Lemma 2.4, we have

$$\prod_{j=1}^k \left( 1 - \lambda_j \left( |A_1A_2|^r \right) \right)$$

$$= \prod_{j=1}^k \left( 1 - \lambda_j \left( |A_1||A_2^*|^r \right) \right)$$

$$\geq \prod_{j=1}^k \left( 1 - \lambda_j \left( |A|^r \left| A^*_r \right| \right) \right)$$

$$\geq \prod_{j=1}^k \left( 1 - \lambda_j \left( |A_1|^{r_1} \right) \frac{1}{p_{r_1}} (1 - \lambda_j \left( |A_2^*|^{r_2} \right) \frac{1}{p_{r_2}} \right)$$

$$= \prod_{j=1}^k \left( 1 - \lambda_j \left( |A_1|^{r_1} \right) \frac{1}{p_{r_1}} (1 - \lambda_j \left( |A_2^*|^{r_2} \right) \frac{1}{p_{r_2}} \right).$$

Suppose (1.4) holds for some $m \geq 2$ and all $r \geq 1$. Let $A_1, \ldots, A_{m+1}$ be contractive and $p_1, \ldots, p_{m+1} > 0$ satisfy $\frac{1}{p_1} + \cdots + \frac{1}{p_{m+1}} = 1$. Set $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_{m}}$. We have

$$\frac{1}{p} + \frac{1}{p_{m+1}} = 1$$  \hspace{1cm} and  \hspace{1cm} $$\frac{1}{p_{1/p}} + \cdots + \frac{1}{p_{m/p}} = 1.$$
For all $1 \leq k \leq n$, we have
\[
\prod_{j=1}^{k} \lambda_{n+1-j} \left( I_n - |A_1 \cdots A_{m+1}|^r \right) \\
\geq \prod_{j=1}^{k} \lambda_{n+1-j} \left( I_n - |A_1 \cdots A_m|^p \right)^{1/p} \lambda_{n+1-j} \left( I_n - |A_{m+1}|^{p_{m+1}} \right)^{1/p_{m+1}} \\
\geq \left( \prod_{i=1}^{m} \prod_{j=1}^{k} \lambda_{n+1-j} \left( I_n - |A_i|^p (p_i/p) \right)^{1/p} \right)^{1/p} \prod_{j=1}^{k} \lambda_{n+1-j} \left( I_n - |A_{m+1}|^{p_{m+1}} \right)^{1/p_{m+1}} \\
= \prod_{i=1}^{m+1} \prod_{j=1}^{k} \lambda_{n+1-j} \left( I_n - |A_i|^{p_i} \right)^{1/p_i}.
\]

Remark 2.5: If we only consider the case when $r = 1$ in Theorem 1.2, the induction argument would not work.

Example 2.6: Let $A = \begin{pmatrix} -1 & 1 \ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & -2 \ 1 & 1 \end{pmatrix}$. Direct computation shows that
\[
\lambda_1 \left( \frac{|A|^2 + |B|^2}{2} \right) = 8.53, \quad \lambda_1(|AB|) = 9.50, \quad \lambda_1(|A^*B|) = 8.86, \quad \lambda_1(|A^*B^*|) = 8.72.
\]

Therefore, the inequality (2.5) may not hold if we replace $AB^*$ by $AB$, $A^*B$ or $A^*B^*$. Consequently, a simple generalization of (2.5) to more than two matrices cannot hold. It would be interesting to know if the inequality (2.6) can be generalized to more than two matrices.

Corollary 2.7: Suppose $A_1, \ldots, A_m \in M_n$ are contractive matrices and $p_1, \ldots, p_m > 0$ satisfy $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$. Then for all $1 \leq k \leq n$, we have
\[
\prod_{j=1}^{k} \lambda_{n+1-j} \left( I_n - (A_1 \cdots A_m) \right) \geq \prod_{i=1}^{m} \prod_{j=1}^{k} \left( 1 - \lambda_j(|A_i|) \right)^{\frac{1}{p_i}}.
\]

Proof: By Lemma 2.1,
\[
\prod_{j=1}^{k} \lambda_{n+1-j} \left( I_n - (A_1 \cdots A_m) \right) \geq \prod_{i=1}^{m} \prod_{j=1}^{k} \sigma_{n+1-j} \left( I_n - (A_1 \cdots A_m) \right) \\
= \prod_{i=1}^{m} \prod_{j=1}^{k} \lambda_{n+1-j} \left( |I_n - (A_1 \cdots A_m)| \right) \\
\geq \prod_{i=1}^{m} \prod_{j=1}^{k} \lambda_{n+1-j} \left( I_n - |A_1 \cdots A_m| \right) \quad \text{by Lemma 2.3} \\
= \prod_{i=1}^{m} \prod_{j=1}^{k} \left( 1 - \lambda_j(|A_1 \cdots A_m|) \right) \\
\geq \prod_{i=1}^{m} \prod_{j=1}^{k} \left( 1 - \lambda_j(|A_i|) \right)^{\frac{1}{p_i}} \quad \text{by Theorem 1.2}.
\]

□
Remark 2.8: Let $A, B \in M_n$ be contractive. Putting $m = 2$, $A_1 = A^*$ and $A_2 = B$ in Corollary 2.7, we have

$$\prod_{j=1}^{k} |\lambda_{n+1-j}(I_n - A^*B)|^2 \geq \prod_{j=1}^{k} \left(1 - \lambda_j(|A|^{p_1})^{\frac{2}{p_1}} \left(1 - \lambda_j(|B|)^{p_2})^{\frac{2}{p_2}}\right)^2 \right. \tag{2.7}$$

for $p_1, p_2 > 0$ with $\frac{1}{p_1} + \frac{1}{p_2} = 1$. In particular, for $p_1 = p_2 = 2$, we have Theorem 1.1.

The next example shows that for some $A$ and $B$, we can have

$$\left(1 - \lambda_j(|A|^{p_1})^{\frac{2}{p_1}} \left(1 - \lambda_j(|B|)^{p_2})^{\frac{2}{p_2}}\right)^2 \right. > \left[1 - \lambda_j(A^*A)]\left[1 - \lambda_j(B^*B)\right] \tag{2.8}$$

for some $p_1$ and $p_2 > 0$ with $\frac{1}{p_1} + \frac{1}{p_2} = 1$ and all $j = 1, \ldots, n$. This shows that (2.7) provides a better bound than (1.2).

Example 2.9: Take

$$A = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

If we take $p_1 = 3$, and $p_2 = \frac{3}{2}$ in (2.8), a simple calculation gives

$$\left[1 - \lambda_1(|A|^{\frac{3}{3}})\right]^{2/3} \left[1 - \lambda_1(|B|^{\frac{3}{2}})\right]^{4/3} = 0.576$$

$$> 0.538 = [1 - \lambda_1(A^*A)]\left[1 - \lambda_1(B^*B)\right],$$

and

$$\left[1 - \lambda_2(|A|^{\frac{3}{3}})\right]^{2/3} \left[1 - \lambda_2(|B|^{\frac{3}{2}})\right]^{4/3} = 0.720$$

$$> 0.683 = [1 - \lambda_2(A^*A)]\left[1 - \lambda_2(B^*B)\right].$$

Hence, the inequality (2.8) holds.

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