The Generalized $K$-Numerical Range

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Let $\mathcal{U}_n$ be the set of all $n \times n$ unitary matrices and $A$ an $n \times n$ normal matrix. For any $1 \leq k \leq n$, denote the $n \times n$ matrix $(a_{ij})$ with $a_{ii} = 1$ for all $1 \leq i \leq k$ and all other entries 0 by $I_{n,k}$. In this note, the author gives a necessary and sufficient condition for the set

$$\mathcal{W}_k(A) = \{\text{diag} U^*AU I_{n,k} : U \in \mathcal{U}_n\}$$

to be convex.

1. INTRODUCTION

Let $\mathcal{U}_n$ be the set of all $n \times n$ unitary matrices and $I_{n,k}$ be the $n \times n$ matrix $(a_{ij})$ with $a_{ii} = 1$ for all $1 \leq i \leq k$ and all other entries zero. We use $A^*$ to denote the conjugate transpose of $A$. For any $n \times n$ complex matrix $A$ and $1 \leq k \leq n$, consider the set $\mathcal{W}_k(A) = \{\text{diag} U^*AU I_{n,k} : U \in \mathcal{U}_n\}$. Clearly, the convexity of $\mathcal{W}_k(A)$ follows from that of the classical field of values of $A$. If $A$ is hermitian, then Horn’s result [2] shows that $\mathcal{W}_n(A)$ is convex. Au-Yeung and Sing [1] proved that if $A$ is normal then $\mathcal{W}_n(A)$ is convex if and only if the eigenvalues of $A$ are collinear. However, the problem for $1 < k < n$, as pointed out by Thompson [3], seems to remain unsolved. In this note, for any $n \times n$ normal matrix $A$ and $1 \leq k \leq n$, we shall give a necessary and sufficient condition for $\mathcal{W}_k(A)$ to be convex.

2. GENERALIZED $K$-NUMERICAL RANGE

Let $A$ be an $n \times n$ complex matrix and $1 \leq k \leq n$. We first observe some elementary properties of $\mathcal{W}_k(A)$.

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Lemma 1  If $\mathcal{W}_k(A)$ is convex, then $\mathcal{W}_r(A)$ is convex for all $1 \leq r \leq k$.

Lemma 2  For any $U \in \mathcal{U}_n$, $\mathcal{W}_k(A) = \mathcal{W}_k(U^* A U)$.

Hence, we can assume a normal matrix to be in diagonal form.

Let $\Omega_n$ denote the set of all $n \times n$ doubly stochastic (d.s.) matrices. A d.s. matrix $(a_{ij})$ is said to be orthostochastic (o.s.) if there exists $(u_{ij}) \in \mathcal{U}_n$ such that $a_{ij} = |u_{ij}|^2$. The set of all $n \times n$ o.s. matrices is denoted by $\mathcal{O}_n$. Let $C^n$ be the set of all complex $n$-tuples. For any $N \in C^n$, $\text{conv}(N)$ is the convex hull of $N$. For simplicity, we just write $C$ for $C^1$.

If $A$ is an $n \times n$ normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, then by Lemma 2, it is easy to see that

$$\mathcal{W}_k(A) = \{ (\lambda_1, \ldots, \lambda_n)(a_{ij})_{I_{n,k}} : (a_{ij}) \in \mathcal{O}_n \}$$

and

$$\text{conv}(\mathcal{W}_k(A)) = \{ (\lambda_1, \ldots, \lambda_n)(d_{ij})_{I_{n,k}} : (d_{ij}) \in \Omega_n \}.$$

If $\lambda_1, \ldots, \lambda_n$ are collinear, then it follows from Horn's result [2] and Lemma 1 that $\mathcal{W}_k(A)$ is convex for all $1 \leq k \leq n$. So, it remains to consider the case when $\lambda_1, \ldots, \lambda_n$ are not collinear. The following results on convex combination are useful in later discussion.

Lemma 3  Let $\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_n \in C$ and $L$ a line on the complex plane passing through $\lambda_1, \ldots, \lambda_r$ such that $\lambda_{r+1}, \ldots, \lambda_n$ lie on one side of an open half plane determined by $L$. A convex combination of the $\lambda_i$'s $\sum_{i=1}^n t_i \lambda_i \in L$ if and only if $t_i = 0$ for all $i \geq r+1$.

Lemma 4  Let $\lambda_1, \ldots, \lambda_r \in C$ be $r$ distinct points which are collinear and lie in the order $\lambda_1, \lambda_2, \ldots, \lambda_r$. If a convex combination $\sum_{i=1}^r t_i \lambda_i = \frac{1}{2}(\lambda_1 + \lambda_2)$, then

i) $t_1 \geq \frac{1}{2}$ and

ii) if $t_1 = \frac{1}{2}$, then $t_2 = \frac{1}{2}$.

Let $\lambda_1, \ldots, \lambda_n \in C$ and $V = \{ v_1, \ldots, v_m \}$ be the set of vertices of $\text{conv}(\lambda_1, \ldots, \lambda_n)$. Clearly $V = \{ \lambda_1, \ldots, \lambda_n \}$. If a vertex $v$ is equal to exactly $r \lambda$, then we say that $v$ is of multiplicity $r$ and write $m(v) = r$.

Theorem 1  Let $A$ be an $n \times n$ normal matrix with non-collinear eigenvalues $\lambda_1, \ldots, \lambda_n$. If $\text{conv}(\lambda_1, \ldots, \lambda_n)$ has a vertex $v$ of multiplicity $r$, then $\mathcal{W}_{r+1}(A)$ is not convex.

Proof  Let $P = \text{conv}(\lambda_1, \ldots, \lambda_n)$. Since $\lambda_1, \ldots, \lambda_n$ are not collinear, $P$ has at least 3 vertices. So, we have $r \leq n - 2$.

After reordering the indices of $\lambda$, we may assume $v = \lambda_1 = \lambda_2 = \cdots = \lambda_r$ and $v \neq \lambda_i$ for all $i \geq r+1$. Consider the two edges $E_1, E_2$ of $P$ meeting at $v$. There may be many $\lambda_i$, with $i \geq r+1$, lying on $E_1$. Choose a $\lambda (\lambda_i, \text{say})$ which is nearest to $\lambda_1$. Therefore, there is no $\lambda$ on $E_1$ lying between $\lambda_1$ and $\lambda_i$. 
Similarly, $\lambda_{i_2}$ is chosen on $E_2$. For $j = 1, 2$, let

$$N_j = \{ I \ni r + 1: \lambda_I = \lambda_{i_j} \}.$$ 

So, we have

$$N_1 \cap N_2 = \emptyset.$$  \hspace{1cm} (1)

Clearly, the point

$$a = (\lambda_{i_1}, \lambda_{i_1}, \ldots, \lambda_{i_1}, \frac{1}{2}(\lambda_{i_1} + \lambda_{i_2}), \frac{1}{2}(\lambda_{i_1} + \lambda_{i_2}), 0, 0, \ldots, 0) \in \text{conv}(\mathcal{W}_r(A)).$$

We are going to show that $a \notin \mathcal{W}_{r+1}(A)$.

Suppose the contrary that $a \in \mathcal{W}_{r+1}(A)$. So, there exists $U = (u_{ij}) \in \mathcal{U}_n$ such that

$$\sum_{i=1}^n |u_{ij}|^2 \lambda_i = \lambda_1 \quad \text{for all } j = 1, \ldots, r-1.$$ 

$$\Rightarrow$$

$$\sum_{i=1}^n |u_{ir}|^2 \lambda_i = \frac{1}{2}(\lambda_{i_1} + \lambda_{i_2})$$

$$\sum_{i=1}^n |u_{r+1}|^2 \lambda_i = \frac{1}{2}(\lambda_{i_1} + \lambda_{i_2})$$

Using the results in Lemma 3 and 4, we have

$$\sum_{i=1}^r |u_{ij}|^2 = 1 \quad \text{for all } j = 1, \ldots, r-1.$$ \hspace{1cm} (2)

$$\sum_{i=1}^r |u_{ir}|^2 \geq \frac{1}{2}$$ \hspace{1cm} (3)

$$\sum_{i=1}^r |u_{r+1}|^2 \geq \frac{1}{2}.$$ \hspace{1cm} (4)

Adding (2), (3) and (4), we have

$$\sum_{j=1}^{r+1} \sum_{i=1}^r |u_{ij}|^2 \geq r.$$ \hspace{1cm} (5)

On the other hand, $U \in \mathcal{U}_n$, therefore

$$\sum_{j=1}^{r+1} \sum_{i=1}^r |u_{ij}|^2 \leq \sum_{i=1}^r \sum_{j=1}^n |u_{ij}|^2 = r.$$ \hspace{1cm} (6)

Combining (5) and (6), we have

$$\sum_{j=1}^{r+1} \sum_{i=1}^r |u_{ij}|^2 = r.$$ \hspace{1cm} (7)

and

$$u_{ij} = 0 \text{ for all } 1 \leq i \leq r \text{ and } r+2 \leq j \leq n.$$ \hspace{1cm} (8)
Also, we must have equality in (3) and (4). That is,

\[ \sum_{i=1}^{r} |u_{ir}|^2 = \frac{1}{2} \]

(9)

and

\[ \sum_{i=1}^{r} |u_{ir+1}|^2 = \frac{1}{2}. \]

(10)

From (9), (10) and the results in Lemma 3 and 4, we have

\[ \sum_{i \in N_1} |u_{ir}|^2 = \sum_{i \in N_2} |u_{ir+1}|^2 = \frac{1}{2} \]

(11)

and hence, for all \( i \leq r + 1 \)

\[ u_{ir} = 0 \text{ for all } i \notin N_1 \]

(12)

and

\[ u_{ir+1} = 0 \text{ for all } i \notin N_2. \]

(13)

Let \( u_j \) be the \( j \)th column of \( U \). Writing each \( u_j = \begin{pmatrix} u_j^{(1)} \\ u_j^{(2)} \end{pmatrix} \), where \( u_j^{(1)} \) and \( u_j^{(2)} \) are of dimension \( r \) and \( n-r \) respectively. From (1), (12) and (13), \( u_j^{(2)} \) is orthogonal to \( u_{r+1}^{(2)} \). Since \( U \in \mathbb{F}_{\kappa} \), \( (u_1, \ldots, u_n) \) is orthonormal. From (8) we have \( u_j^{(1)} = 0 \) for all \( j > r + 1 \). Therefore

\( (\sqrt{2}u_r^{(2)}, \sqrt{2}u_{r+1}^{(2)}, u_{r+2}^{(2)}, \ldots, u_n^{(2)}) \)

is an orthonormal family. But each \( u_j^{(2)} \) is an \( n-r \) dimensional vector and there are \( n-r+1 \) of them. This leads to contradiction. Hence, \( a \notin \mathbb{F}_{r+1}(A) \) and consequently \( \mathbb{F}_{r+1}(A) \) is not convex.

**Theorem 2** Let \( A \) be an \( n \times n \) normal matrix with non-collinear eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \). Let \( V = \{v_1, \ldots, v_m\} \) be the set of vertices of \( \text{conv}(\lambda_1, \ldots, \lambda_n) \) and \( r = \text{Min}\{m(v_i) : v_i \in V\} \). For any \( 1 \leq k \leq n \), \( \mathbb{F}_k(A) \) is convex if and only if \( k \leq r \).

**Proof** If \( k > r \), then take any vertex with multiplicity \( r \) and by Theorem 1, \( \mathbb{F}_{r+1}(A) \) is not convex. So by Lemma 1, \( \mathbb{F}_k(A) \) is not convex.

If \( k \leq r \), it suffices to prove that \( \mathbb{F}_k(A) \) is convex. It is easy to see that \( mr \leq n \) and with suitable rearrangement of \( \lambda \), we may assume

\[ v_1 = \lambda_1 = \cdots = \lambda_r, \]
\[ v_2 = \lambda_{r+1} = \cdots = \lambda_{2r}, \]
\[ \vdots \]
\[ v_m = \lambda_{(m-1)r+1} = \cdots = \lambda_{mr}. \]
For any $a = (a_1, ..., a_r, 0, ..., 0) \in \text{conv}(\mathcal{W}_r(A))$, clearly $a_j \in \text{conv}(v_1, ..., v_m)$ for all $1 \leq j \leq r$. Let

$$a_j = \sum_{i=1}^{m} d_{ij} v_i$$

where $d_{ij} \geq 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq r$, and $\sum_{i=1}^{m} d_{ij} = 1$, for all $1 \leq j \leq r$. For all $1 \leq i \leq n$ and $1 \leq j \leq r$, define

$$b_{ij} = \begin{cases} d_{ij} & \text{if } i = (t-1)r + j \text{ for some } 1 \leq t \leq m \\ 0 & \text{otherwise.} \end{cases}$$

That is $(b_{ij})$ is of the form

$$
\begin{bmatrix}
  d_{11} & 0 & \cdots & 0 \\
  d_{12} & d_{1r} \\
  0 & \ddots & \ddots & \vdots \\
  d_{r1} & 0 & \ddots & d_{rr} \\
  d_{m1} & \cdots & 0 & d_{mr} \\
  0 & \cdots & \cdots & 0 \\
\end{bmatrix}
$$

Define $r$ column vectors $u_1, ..., u_r$ by

$$u_j = \left( \sqrt{b_{1j}}, \sqrt{b_{2j}}, ..., \sqrt{b_{nj}} \right)$$

It is clear that $(u_1, ..., u_r)$ is orthonormal. So, we can extend it to an orthonormal base $(u_1, ..., u_r, u_{r+1}, ..., u_n)$ of $\mathbb{C}^n$ and thus $(u_{ij})$, which is formed with the $j$th column as $u_j$, is in $\mathcal{W}_n$. It follows from the construction that

$$(\lambda_1, ..., \lambda_n)(|u_{ij}|^2)I_{n,r} = a.$$ 

Therefore $a \in \mathcal{W}_r(A)$. Hence, $\mathcal{W}_r(A)$ is convex. Furthermore, we can easily see from the above construction that in this case

$$\mathcal{W}_r(A) = \{(a_1, ..., a_r, 0, ..., 0) : a_i \in \text{conv}(\lambda_1, ..., \lambda_n)\}.$$
COROLLARY 1 If $A$ is a normal matrix with noncollinear eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $\text{conv}(\lambda_1, \ldots, \lambda_n)$ has $m$ vertices, then $\mathcal{W}_k(A)$ is not convex for all $k \geq (n+1)/m$.

In particular, if $\lambda_1, \ldots, \lambda_n$ are not collinear then $\text{conv}(\lambda_1, \ldots, \lambda_n)$ has at least 3 vertices. By making use of Horn's result [2], we have

COROLLARY 2 Let $A$ be a normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. If $\mathcal{W}_k(A)$ is convex for some $k \geq (n+1)/3$, then $\mathcal{W}_n(A)$ is convex.

COROLLARY 3 Let $B = \text{diag}(\lambda_1, \ldots, \lambda_n, \lambda_{n+1})$ and $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$. If $\mathcal{W}_{k+1}(B)$ is convex, then $\mathcal{W}_k(A)$ is also convex.

References