Mean curvature flow of entire graphs evolving away from the heat flow

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August 19, 2016

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Introduction
We consider an initial graph $u_0 : \mathbb{R}^n \to \mathbb{R}$ with $\|u_0\|_{C^{2,\alpha}} \leq C$.

**Heat Flow**

\[
\frac{\partial v}{\partial t} = \Delta v
\]

$v(x, 0) = u_0(x)$

**Mean Curvature Flow**

\[
\frac{\partial u}{\partial t} = \sqrt{1 + |Du|^2} \, \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right)
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### Heat Flow

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Long-time existence

### Mean Curvature Flow

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Long-time existence [Ecker-Huisken '84]
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### Heat Flow
\[ \frac{\partial v}{\partial t} = \Delta v \]
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Long-time existence

Maximum principle

### Mean Curvature Flow
\[ \frac{\partial u}{\partial t} = \sqrt{1 + |Du|^2} \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) \]
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Long-time existence [Ecker-Huisken ’84]

Disjoint surfaces stay disjoint
Dimension \( n = 1 \)

Heat flow

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Mean curvature flow

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Nara-Taniguchi [’07] proved that the solution to the MCF tends to the solution to the heat flow (\( n = 1 \)):

\[
\sup_{x \in \mathbb{R}} |u(x, t) - v(x, t)| \leq C t^{-1/2}, \quad t > 0
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Dimension $n = 1$

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\[ \frac{\partial u}{\partial t} = u_{xx} - (u_x - \arctan u_x)_x \]

\[ \frac{\partial u}{\partial t} = u_{xx} + F_x \]
Dimension $n = 1$

Heat flow

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Mean curvature flow

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Proposition (Repnikov-Eidelman ’67)

Let \( v(x, t) \) be a solution to the heat flow. The limit

\[
\lim_{t \to \infty} v(x, t) = A(x)
\]

exists uniformly in \( x \in \mathbb{R}^n \) if and only if

\[
\lim_{\rho \to \infty} \frac{1}{|B_{x, \rho}|} \int_{B_{x, \rho}} u_0(y) \, dy = A(x)
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exists uniformly in \( x \in \mathbb{R}^n \); Moreover \( A(x) \equiv C \).
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Question: Is the Nara-Taniguchi result true for $n \geq 2$?
Stabilization of the heat flow

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- Yes for $u_0$ rotationally symmetric [Nara ’08].
### Proposition (Repnikov-Eidelman ’67)

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**Question:** Is the Nara-Taniguchi result true for \( n \geq 2 \)?

- Yes for \( u_0 \) rotationally symmetric [Nara ’08].
- No in general [Drugan - N. ’16]
Heat flow stabilizes; MCF stabilizes
Initial graph $U_0$

Place tall spikes of volume 1 on $\mathbb{Z}^n$
Place tall spikes of volume 1 on $\mathbb{Z}^n$

**Heat flow**

Because the average over larger balls tends to 1,

$$\lim_{t \to \infty} v(\cdot, t) \to 1.$$
A useful fact

There are self-shrinking doughnuts. Choose a scaled version $\mathbb{T}_0$ so that

$$r_0 < \frac{1}{2}$$

We denote its extinction time $t_*$. 

Figure 1: A cross section of $\mathbb{T}_0$. 

Initial graph $\mathcal{U}_0$

Place tall spikes of volume 1 on $\mathbb{Z}^n$
Place tall spikes of volume 1 on $\mathbb{Z}^n$

**Mean curvature flow**

$$u(\cdot, t) \leq \delta_0, \quad t \geq t^*. $$
Mean curvature flow

\[ u(\cdot, t) \leq \delta_0, \quad t \geq t_*. \]
Mean curvature flow

\[ u(\cdot, t) \leq \delta_0, \quad t \geq t^*. \]

Because of interior bounds by Ecker-Huisken ['91],

\[ \lim_{t \to \infty} u(\cdot, t) = C \leq \delta_0 \]
MCF oscillates; Heat flow stabilizes
A solution that oscillates indefinitely

Let $I_n = (n! + 1, n! - 1)$, and

\[ u_0 = \begin{cases} 
1, & |x| \in I_{2k+1} \\
0, & |x| \in I_{2k} \end{cases} \]

We have

\[ \liminf_{t \to \infty} \nu(0, t) = 0, \quad \limsup_{t \to \infty} \nu(0, t) = 1 \]
A solution that oscillates indefinitely

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$$u_0 = \begin{cases} 
1, & |x| \in l_{2k+1} \\
0, & |x| \in l_{2k}
\end{cases}$$

We have

$$\lim_{t \to \infty} \inf v(0, t) = 0, \quad \lim_{t \to \infty} \sup v(0, t) = 1$$
An initial graph $\mathcal{W}_0$

Heat flow: uniform convergence to $\lim_{t \to \infty} v(\cdot, t) = 1$. 

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An initial graph $\mathcal{W}_0$

**Heat flow: uniform convergence to 1**

$$\lim_{t \to \infty} v(\cdot, t) = 1.$$
An initial graph $w_0$

$w_0$

Mean curvature flow: barriers

- $u(\cdot, t^*) \leq r_0$ on the odd slabs.
- $u(\cdot, t^*) \leq 1 + r_0$ on the even slabs.
An initial graph $\mathcal{W}_0$

Mean curvature flow: barriers

- $u(\cdot, t_*) \leq r_0$ on the odd slabs.
- $u(\cdot, t_*) \leq 1 + r_0$ on the even slabs.
An initial graph $\psi_0$

Spikes here.

- $\psi_0$ is a lower barrier.
- $\psi_0 + r_0$ is an upper barrier after time $t^\ast$. 

$S_{2k}$, $S_{2k+1}$, $S_{2k+2}$

$(2k)!$, $(2k + 1)!$, $(2k + 2)!$
An initial graph $\psi_0$

Mean curvature flow: the solution oscillates

- $\psi_0$ is a lower barrier.

$$\limsup_{t \to \infty} u(0, t) \geq 1$$
Mean curvature flow: the solution oscillates

- $\psi_0$ is a lower barrier.
- “$\psi_0 + r_0$” is an upper barrier after time $t_*$. 

$$\limsup_{t \to \infty} u(0, t) \geq 1$$
Mean curvature flow: the solution oscillates

- $\psi_0$ is a lower barrier.
- "$\psi_0 + r_0$" is an upper barrier after time $t_*$.

$$\limsup_{t \to \infty} u(0, t) \geq 1$$

$$\liminf_{t \to \infty} u(0, t) \leq r_0 \leq \varepsilon$$
Thank you for your attention!