An asymptotic multipartite Kühn-Osthus theorem

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The Hajnal-Szemerédi theorem

Theorem (Hajnal-Szemerédi, 1970)

(Complementary form) If $G$ is a simple graph on $n$ vertices with minimum degree

$$\delta(G) \geq \left(1 - \frac{1}{k}\right)n$$

then $G$ contains a subgraph which consists of $\lfloor n/k \rfloor$ vertex-disjoint copies of $K_k$. 

Notes

$k = 2$ follows from Dirac $k = 3$ proven by Corrádi & Hajnal 1963
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### Theorem (Hajnal-Szemerédi, 1970)

*(Complementary form)* If $G$ is a simple graph on $n$ vertices with minimum degree $\delta(G) \geq \left(1 - \frac{1}{k}\right)n$, then $G$ contains a subgraph which consists of $\left\lfloor \frac{n}{k} \right\rfloor$ vertex-disjoint copies of $K_k$.

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- $k = 2$ follows from Dirac
- $k = 3$ proven by Corrádi & Hajnal 1963
The Alon-Yuster theorem

**Theorem (Alon-Yuster, 1992)**

For any $\alpha > 0$ and graph $H$, there exists an $n_0 = n_0(\alpha, H)$ such that in any graph $G$ on $n \geq n_0$ vertices with

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right)n + \alpha n$$

there is an $H$-tiling of $G$ if $|V(H)|$ divides $n$.

**Theorem (Kühn-Osthus, 2009)**

For any graph $H$, there exists an $n_0 = n_0(H)$ and a constant $C = C(H)$ such that in any graph $G$ on $n \geq n_0$ vertices with

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there is an $H$-tiling of $G$ if $|V(H)|$ divides $n$.

This result is best possible, up to the constant $C$.

But what is $\chi^*(H)$?
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Komlós, Sárközy and Szemerédi, 2001, showed that $\alpha n$ can be replaced by $C = C(H)$, but not eliminated entirely.
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Martin (Iowa State University University of Birmingham London School of Economics)
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But what is $\chi^*(H)$?
Critical chromatic number

Definition

Let $H$ be a graph with

- order: $h = |V(H)|$
- chromatic number: $\chi = \chi(H)$
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**Fact**

For any graph $H$:

$$\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$$
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The critical chromatic number of \( H \), is \( \chi_{cr} = \chi_{cr}(H) = \frac{(\chi-1)h}{h-\sigma} \)

Fact

For any graph \( H \):

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\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)
\]

Also, \( \chi_{cr}(H) = \chi(H) \) iff every proper \( \chi \)-coloring of \( H \) is a equipartition.

\( \chi_{cr}(H) \) was defined by Komlós, 2000.
Critical chromatic number

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The critical chromatic number of $H$, is $\chi_{cr} = \chi_{cr}(H) = \frac{(\chi-1)h}{h-\sigma}$

$$\chi^*(H) = \begin{cases} \chi_{cr}(H), & \text{if } \gcd(H) = 1; \\ \chi(H), & \text{else.} \end{cases}$$

where $\gcd(H)$ is basically the gcd of the differences of the color classes in proper colorings of $H$. 

Definitions

Definition
The family of \( k \)-partite graphs with \( n \) vertices in each part is denoted \( \mathcal{G}_k(n) \).

Definition
The natural bipartite subgraphs of \( G \) are the ones induced by the pairs of classes of the \( k \)-partition.

Definition
If \( G \in \mathcal{G}_k(n) \), let \( \hat{\delta}_k(G) \) denote the minimum degree among all of the natural bipartite subgraphs of \( G \).
Multipartite Hajnal-Szemerédi

The asymptotic Hajnal-Szemerédi theorem was solved with two different methods:

Theorem (Keevash-Mycroft, 2013; Lo-Markström, 2013)

Let $k \geq 2$ and $\epsilon > 0$. There exists an $n_0 = n_0(k, \epsilon)$ such that if $n \geq n_0$, $G \in G_k(n)$ and if

$$\hat{\delta}_k(G) \geq \left(1 - \frac{1}{k}\right) n + \epsilon n,$$

then $G$ has a $K_k$-tiling.

Hypergraph blow-up; Absorbing method
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then $G$ has a $K_k$-tiling.

Hypergraph blow-up; Absorbing method
In a longer manuscript, Keevash and Mycroft settle the multipartite Hajnal-Szemerédi case for large $n$:


Let $k \geq 2$. There exists an $n_0 = n_0(k)$ such that if $n \geq n_0$, $G \in G_k(n)$ and if

$$\hat{\delta}_k(G) \geq \left(1 - \frac{1}{k}\right)n,$$

then $G$ has a $K_k$-tiling or both $k$ and $n/k$ are odd integers and $G \approx \Gamma_k(n/k)$.

The case of $k = 3$ was solved by Magyar-M. (2002). The case of $k = 4$ was solved by M.-Szemerédi (2008).

The graph $\Gamma_k(n/k)$ is one of Catlin’s “Type 2” graphs.
Catlin’s Type 2 Graphs

Catlin’s Type 2 graph.

The red indicates non-edges between graph classes.
Theorem (Zhao, 2009)

Let \( h \) be a positive integer. There exists an \( n_0 = n_0(h) \) such that if \( n \geq n_0, h \mid n \), and \( G \in \mathcal{G}_2(n) \) with

\[
\delta(G) = \delta_2(G) \geq \begin{cases} 
\frac{1}{2}n + h - 1, & \text{if } n/h \text{ is odd;} \\
\frac{1}{2}n + \frac{3h}{2} - 2, & \text{if } n/h \text{ is even},
\end{cases}
\]

then \( G \) has a perfect \( K_{h,h} \)-tiling.

Moreover, there are examples that prove that this \( \delta_2 \) condition cannot be improved.
Toward Kühn-Osthus

Theorem (Bush-Zhao, 2012)

Let \( H \) be a bipartite graph. There exists an \( n_0 = n_0(H) \) and \( c = c(H) \) such that if \( n \geq n_0 \), \( |V(H)| | n \), and \( G \in \mathcal{G}_2(n) \) with

\[
\delta(G) \geq \begin{cases} 
\left(1 - \frac{1}{\chi^*(H)}\right)n + c, & \text{if } \gcd(H) = 1 \text{ or } \gcd_{cc}(H) > 1; \\
\left(1 - \frac{1}{\chi(H)}\right)n + c, & \text{if } \gcd(H) > 1 \text{ and } \gcd_{cc}(H) = 1,
\end{cases}
\]

then \( G \) has a perfect \( H \)-tiling.

The quantity \( \gcd_{cc}(H) \) counts the gcd of the sizes of the connected components of \( H \).
Our results

Theorem (M.-Skokan, 2013+)

Let $k \geq 2$, $H$ be a graph with $\chi(H) = k$ and $\epsilon > 0$. There exists an $n_0 = n_0(H, \epsilon)$ such that if $n \geq n_0$, $G \in G_k(n)$ and if

$$\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi(H)}\right)n + \epsilon n,$$

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This, of course, contains the asymptotic Hajnal-Szemerédi case.
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Our results

**Theorem (M.-Mycroft-Skokan, 2015+)**

Let \( k \geq 2, \ H \) be a graph with \( \chi(H) = k, \ \chi^* = \chi^*(H) \) and \( \epsilon > 0 \). There exists an \( n_0 = n_0(H, \epsilon) \) such that if \( n \geq n_0, \ G \in G_k(n) \) and if

\[
\hat{\delta}_k(G) \geq \left( 1 - \frac{1}{\chi^*(H)} \right) n + \epsilon n,
\]

then \( G \) has an \( H \)-tiling.

The main tool is linear programming.
Linear programming

Definition

For any graph $G$, let $\mathcal{T}_k(G)$ denote the set of $k$-cliques of $G$. The \textbf{FRACTIONAL $K_k$-TILING NUMBER}, $\tau^*_k(G)$ is:

$$\tau^*_k(G) = \begin{cases} 
\max & \sum_{T \in \mathcal{T}_k(G)} w(T) \\
\text{s.t.} & \sum_{T \in \mathcal{T}_k(G), T \ni v} w(T) \leq 1, \quad \forall v \in V(G), \\
& w(T) \geq 0, \quad \forall T \in \mathcal{T}_k(G). 
\end{cases}$$
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Theorem

Let $k \geq 2$. If $G \in \mathcal{G}_k(n)$ and $\delta_k(G) \geq (k - 1)n/k$, then $\tau^*_k(G) = n$. 

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Let \( k \geq 2 \). If \( G \in \mathcal{G}_k(n) \) and \( \hat{\delta}_k(G) \geq (k - 1)n/k \), then \( \tau^*_k(G) = n \).

The proof is by induction on \( k \) and uses both the Duality Theorem and Complementary Slackness Theorem of LPs.
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**Theorem**

Let $k \geq 2$. If $G \in G_k(n)$ and $\delta_k(G) \geq (k - 1)n/k$, then $\tau_k^*(G) = n$.

The proof is by induction on $k$ and uses both the Duality Theorem and Complementary Slackness Theorem of LPs.

**Duality Theorem:**

$$\tau_k^*(G) = \begin{cases} \max & \sum w(T) \\ \text{s.t.} & \sum_{T \ni v} w(T) \leq 1, \ \forall v, \\ w(T) \geq 0, \quad \forall T. \end{cases} = \begin{cases} \min & \sum x(v) \\ \text{s.t.} & \sum_{v \in T} x(v) \geq 1, \ \forall T, \\ x(v) \geq 0, \quad \forall v. \end{cases}$$
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UB: \( \tau^*_k(G) \leq n \).

Setting \( x(v) \equiv 1/k \) gives a feasible solution to the minLP, so \( \tau^*_k(G) \leq (kn) \cdot (1/k) = n \).
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**Theorem**

Let $k \geq 2$. If $G \in \mathcal{G}_k(n)$ and $\hat{\delta}_k(G) \geq (k - 1)n/k$, then $\tau_k^*(G) = n$. 

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**LB:** $\tau_k^*(G) \geq n$. Base Case: $k = 2$. 

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Let \( G = (V_1, V_2; E) \). If either \( V_1 \) or \( V_2 \) fails to have a “slack vertex” in the maxLP, then

\[
\tau_k^*(G) \geq \sum_T w(T) = \sum_{v \in V_i} \sum_{T \ni v} w(T) = \sum_{v \in V_i} 1 = n.
\]
Let $G = (V, E)$. If either $V_1$ or $V_2$ fails to have a “slack vertex” in the maxLP, then

$$\tau^*_k(G) \geq \sum_{T} w(T) = \sum_{T \ni v} \sum_{T \ni v} w(T) = \sum_{v \in V} 1 = n.$$ 

If $v_1 \in V_1$ and $v_2 \in V_2$ are slack, then we may assume $x(v_1) = x(v_2) = 0$ (Complementary Slackness).
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If \( v_1 \in V_1 \) and \( v_2 \in V_2 \) are slack, then we may assume \( x(v_1) = x(v_2) = 0 \) (Complementary Slackness).

Each vertex in \( N(v_1), N(v_2) \) has weight 1. Since \( |N(v_1)|, |N(v_2)| \geq n/2 \), \( \tau^*_k(G) \geq n \).
Linear programming

\[ \tau_k^*(G) = \begin{cases} \max & \sum w(T) \\ s.t. & \sum_{T \ni v} w(T) \leq 1, \quad \forall v, \\ w(T) \geq 0, \quad \forall T. \end{cases} = \begin{cases} \min & \sum x(v) \\ s.t. & \sum_{v \in T} x(v) \geq 1, \quad \forall T, \\ x(v) \geq 0, \quad \forall v. \end{cases} \]

**LB:** \( \tau_k^*(G) \geq n. \) Induction Step

Let \( G = (V_1, \ldots, V_k; E) \). If any \( V_i \) has no slack vertices in the maxLP, then

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Let $G = (V_1, \ldots, V_k; E)$. If any $V_i$ has no slack vertices in the maxLP, then

$$\tau_k^*(G) \geq \sum_T w(T) = \sum_{v \in V_i} \sum_{T \ni v} w(T) = \sum_{v \in V_i} 1 = n.$$  

If $v_i \in V_i, \forall i$, are slack, then we may assume $x(v_i) = 0, \forall i$.

Let $G_i \leq G[N(v_i)], \forall i$, so that $G_i$ has exactly $\frac{k-1}{k}n$ vertices in each $V_j$. 

Martin (Iowa State University  University of Birmingham  London School of Economics) 
An asymptotic multipartite Kühn-Osthus theorem  08 August 2017  11 / 13
**Linear programming**

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By induction,

$$(k - 1)\tau^*_k(G) \geq \sum_{i=1}^{k} \sum_{v \in V(G_i)} x(v) \geq \sum_{i=1}^{k} \frac{k - 1}{k}n = (k - 1)n.$$
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□
Future work

Can we replace \( \hat{\delta}_k(G) \geq (1 - \frac{1}{\chi^*(G)}) n + \epsilon n \) with \( \hat{\delta}_k(G) \geq (1 - \frac{1}{\chi^*(G)}) n + C(H) \)?
Future work

- Can we replace $\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi^*(G)}\right) n + \epsilon n$ with $\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi^*(G)}\right) n + C(H)$?

- Is $\hat{\delta}_k(G) \geq (k - 1)n/k + \epsilon n$ sufficient to force the $k^{th}$ power of a Hamilton cycle? (Related to Bollobás-Komlós conjecture on bandwidth)
Can we replace $\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi^*(G)}\right) n + \epsilon n$ with $\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi^*(G)}\right) n + C(H)$?

Is $\hat{\delta}_k(G) \geq (k - 1)n/k + \epsilon n$ sufficient to force the $k^{th}$ power of a Hamilton cycle? (Related to Bollobás-Komlós conjecture on bandwidth)

What probability $p$ guarantees that, for any $G$ with $\hat{\delta}_k(G) \geq (k - 1)pn/k + \epsilon pn$, the random subgraph $G_p$ has a $K_k$-tiling?
Thanks!

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http://orion.math.iastate.edu/rymartin

My CV (with links to this and previous talks):

http://orion.math.iastate.edu/rymartin/cv/RMcv.pdf

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