Planar Turán number of the 6-Cycle

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Joint work with:
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Mantel's Theorem

Definition

The maximum number of edges in n-vertex, H-free graph is

$$\mathrm{ex}(n, H).$$

Fact

An n-vertex graph has $\leq \binom{n}{2}$ *edges.*

Theorem (Mantel, 1907)

An n-vertex K_3*-free graph has* $\leq \left\lfloor \frac{n^2}{4} \right\rfloor$ *edges.*

Mantel's Theorem

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The maximum number of edges in n-vertex, H-free planar graph is

$$\mathrm{ex}_{\mathcal{P}}(n, H).$$

Theorem (via Euler, 1758)

An n-vertex planar graph has $\leq 3n - 6$ *edges or it is* K_2.

Theorem (via Euler, 1758)

An n-vertex K_3*-free planar graph has* $\leq 2n - 4$ *edges or it is* K_2.

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$$2e = \sum_{\phi \in F(G)} \deg(\phi) \geq f \cdot 4 = (e - n + 2) \cdot 4$$

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$$2(n-2) \geq e.$$

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Turán's Theorem

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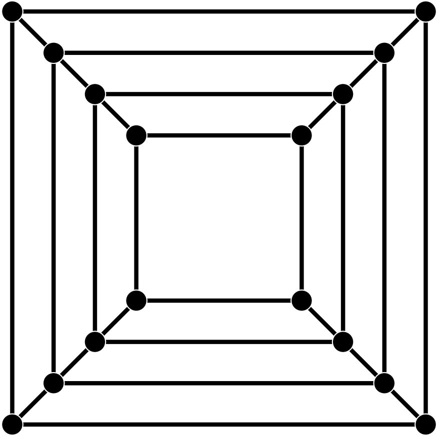
$$\mathrm{ex}_{\mathcal{P}}(n, K_3) = 2(n-2), \qquad \textit{for all } n \geq 3.$$



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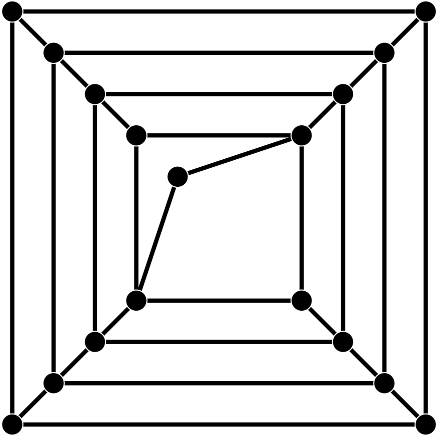

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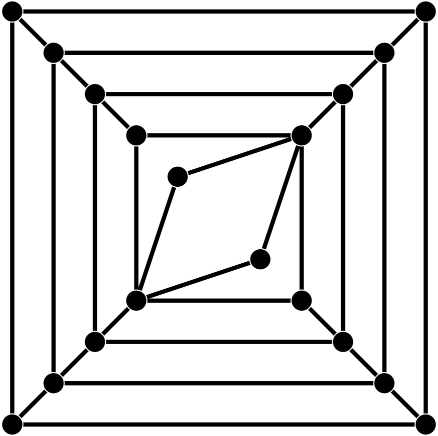

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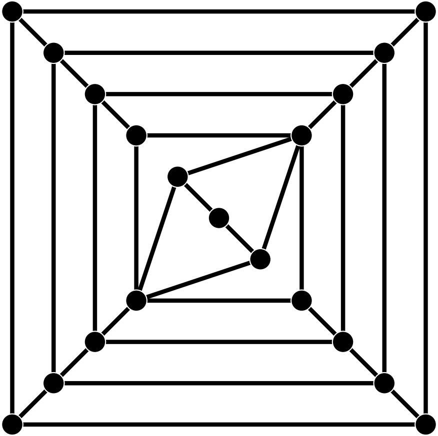

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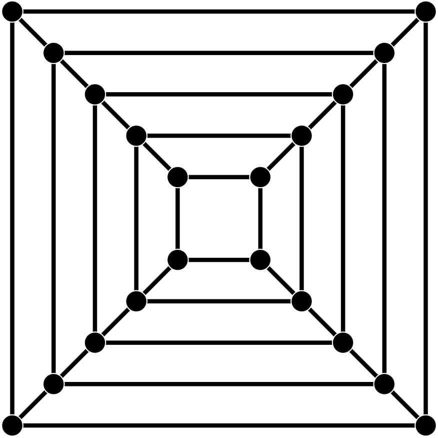

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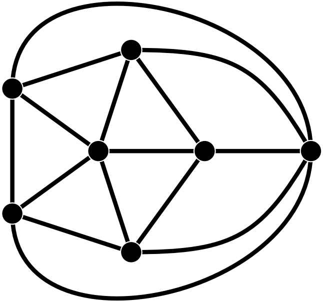

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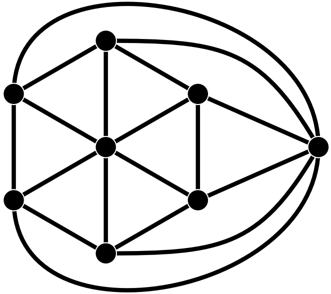

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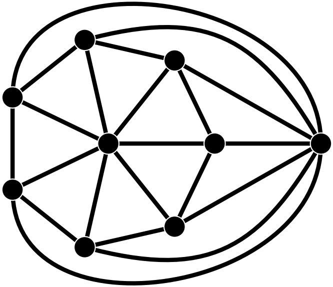

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$$\mathrm{ex}_{\mathcal{P}}(n, K_5) = 3(n-2), \qquad \textit{for all } n \geq 3.$$

Cycle results

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$$\mathrm{ex}_{\mathcal{P}}(n, C_3) = 2(n-2) \qquad \textit{for all } n \geq 3.$$

Theorem (Dowden, 2016)

$$\mathrm{ex}_{\mathcal{P}}(n, C_4) \leq \frac{15}{7}(n-2), \qquad \textit{for all } n \geq 4.$$

Equality holds for all $n \cong 30 \pmod{70}$.

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Equality holds for an infinite sequence of n *for which* $n \cong 9 \pmod{15}$.
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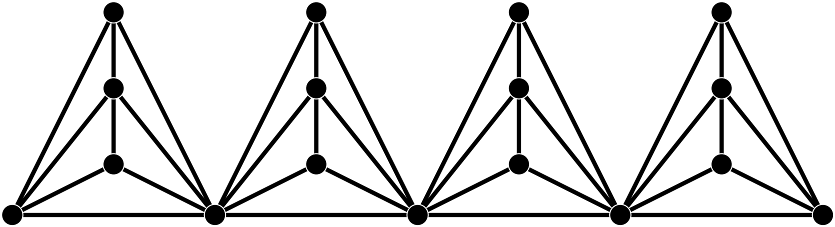
The proof:

1. Why is $n \geq 18$ necessary?
2. Which construction gives $e(G) = \frac{5}{2}n - 7$ if $n \cong 10 \pmod{18}$?
3. How do we establish that planar, C_6-free implies $e(G) \leq \frac{5}{2}n - 7$?

(i) Construction for $n = 17$



$$|V(G)| = 17$$

$$|E(G)| = 36 > 35.5 = \frac{5}{2} \cdot 17 - 7$$

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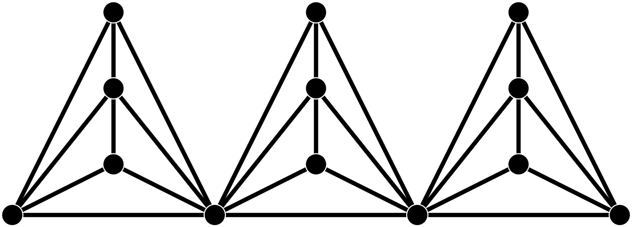

$$|V(G)| = 13$$

$$|E(G)| = 27 > 25.5 = \frac{5}{2} \cdot 16 - 7$$

(ii) Constructions for $n \cong 10 \pmod{18}$: Underlying graph

Theorem

A n-vertex ($n \geq 4$) girth-g planar graph has $\leq \max\left\{\frac{g}{g-2}(n-2), n-1\right\}$ *edges.*

Lemma

For every $k \geq 0$, there is a girth-7 graph G_0^k such that

$$n = 10k + 7 \qquad n_2 = 2k + 7$$

$$e = 14k + 7 = \frac{7}{5}(n-2) \qquad n_3 = 8k.$$

For every $k \geq 1$, there is a girth-7 graph H_0^k such that

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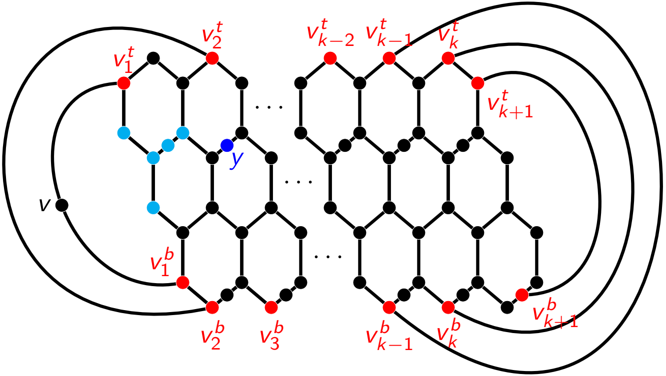

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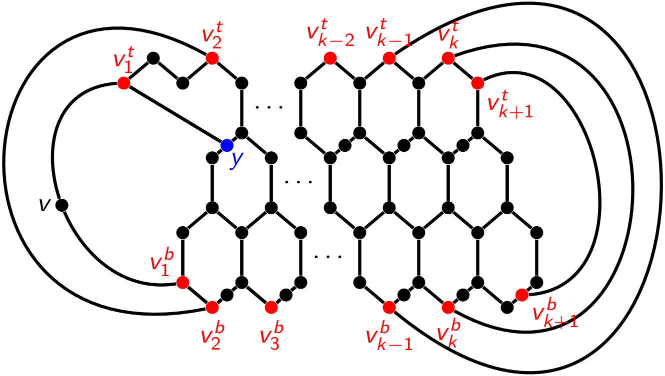


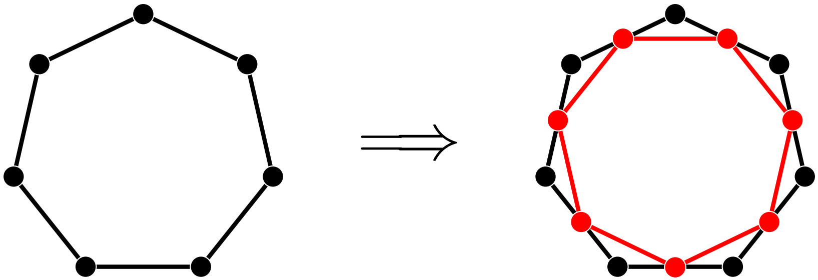
$k \geq 2$

(ii) Constructions for $n \cong 10 \pmod{18}$: Construction

Given

A girth-7 graph, G_0 with vertices of degree 2 and 3.

(1) For every edge, add a "halving vertex":



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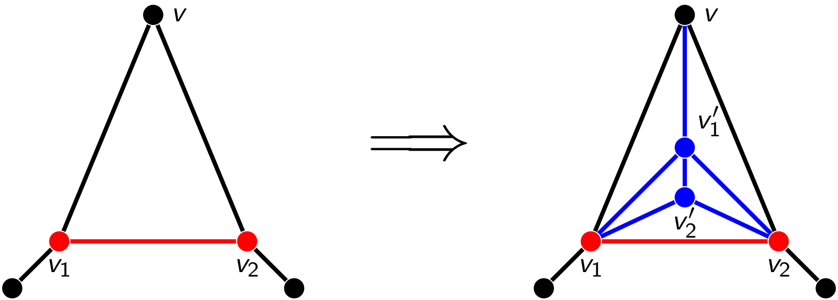
(2) For every degree-2 vertex, v, replace it with a K_5^- as follows:



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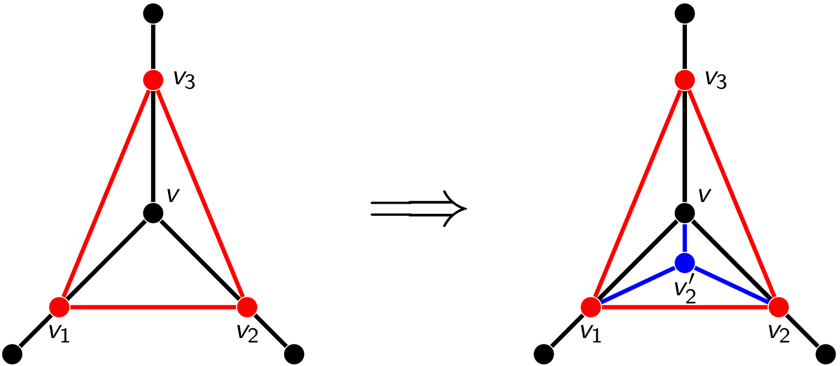

(ii) Constructions for $n \cong 10 \pmod{18}$: Construction

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Let G_0 be chosen such that

$$
\begin{aligned}
v(G_0) & &= v \\
e(G_0) &= \frac{7}{5}(v(G_0) - 2) &= \frac{7v - 14}{5}
\end{aligned}
$$

The resulting graph G has the following:

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\begin{aligned}
v(G) &= v(G_0) + e(G_0) + 2n_2(G_0) + n_3(G_0) \\
&= v + \frac{7v - 14}{5} + 2\left(\frac{v + 28}{5}\right) + \frac{4v - 28}{5} &= \frac{18v + 14}{5} \\
e(G) &= 9v(G_0) &= 9v.
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$$= v + \frac{7v - 14}{5} + 2\left(\frac{v + 28}{5}\right) + \frac{4v - 28}{5} \qquad = \frac{18v + 14}{5}$$
$$e(G) = 9v(G_0) \qquad = 9v.$$

Hence,

$$e(G) = 9\left(\frac{5v(G) - 14}{18}\right) \qquad = \frac{5}{2}v(G) - 7.$$

Moreover, if $v \cong 2 \pmod{5}$, then $v(G) \cong 10 \pmod{18}$.

(iii) Proof for upper bound: Triangular-blocks

Theorem (Ghosh-Győri-M-Paulos-Xiao, 2020+)

$$\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{5}{2}n - 7, \qquad \textit{for all } n \geq 18.$$

We may assume:
- No vertex degree 2.
- No cut-vertex.

KEY IDEA: Partition the edges into *triangular-blocks*:

Definition

For a plane graph G, a triangular-block H, is a set of edges defined as:

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For each triangular-block $B \in \mathcal{B}$:

- $e(B)$ is the number of edges in B.
- $n(B)$ is the number of vertices in B.
  If a vertex is in d blocks, it assigns $1/d$ to each block.
- $f(B)$ is the number of faces in B.
  If a face is incident to d blocks, it assigns $1/d$ to each edge...
  ...unless it has two edges of a K_5^-, then more complicated.

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If G has n vertices, e edges and f faces, then

$$2e - 5n + 14 = 7f + 2n - 5e = \sum_{B \in \mathcal{B}} (7f(B) + 2n(B) - 5e(B)).$$

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$$2e - 5n + 14 = 7f + 2n - 5e = \sum_{B \in \mathcal{B}} \left(7f(B) + 2n(B) - 5e(B)\right).$$

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Because G is C_6-free, every triangular-block has at most 5 vertices.

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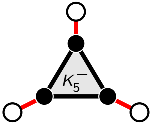
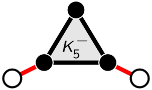
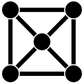
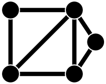
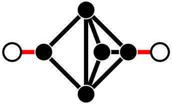
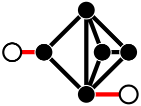
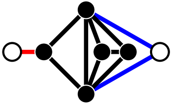
B	Diagram	$f(B) \leq$	$n(B) \leq$	$e(B) =$	$7f + 2n - 5e \leq$
$B_{5,a}$		$5 + \frac{3}{7}$	$2 + \frac{3}{2}$	9	0
$B_{5,a}$		$5 + \frac{2}{7}$	$3 + \frac{2}{2}$	9	0
$B_{5,b}$		$4 + \frac{4}{7}$	$3 + \frac{2}{2}$	8	0

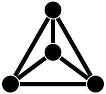
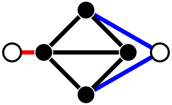
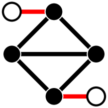
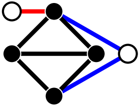
Red edges do not belong to the block but indicate incidence.

(iii) Proof for upper bound: Triangular-blocks

B	Diagram	$f(B) \leq$	$n(B) \leq$	$e(B) =$	$7f + 2n - 5e \leq$
$B_{5,c}$		$3 + \frac{5}{7}$	$3 + \frac{2}{2}$	7	-1
$B_{5,d}$		$4 + \frac{4}{7}$	$3 + \frac{2}{2}$	8	0
$B_{5,d}$		$4 + \frac{4}{7}$	$3 + \frac{2}{2}$	8	0
$B_{5,d}$		$4 + \frac{2}{4} + \frac{2}{7}$	$2 + \frac{3}{2}$	8	$\frac{1}{2}$ ⋆

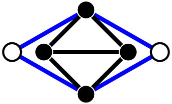
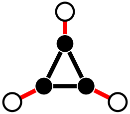
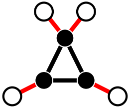
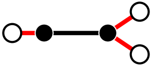
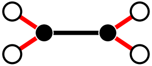
Red edges do not belong to the block but indicate incidence.
Blue edges do not belong to the block and indicate an adjacent 4-face.

(iii) Proof for upper bound: Triangular-blocks

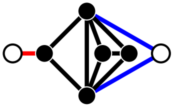
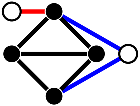
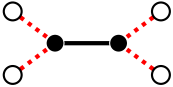
B	Diagram	$f(B) \leq$	$n(B) \leq$	$e(B) =$	$7f + 2n - 5e \leq$
$B_{4,a}$		$3 + \frac{3}{7}$	$2 + \frac{2}{2}$	6	0
$B_{4,b}$		$2 + \frac{2}{4} + \frac{2}{7}$	$1 + \frac{3}{2}$	5	$-\frac{1}{2}$
$B_{4,b}$		$2 + \frac{4}{7}$	$2 + \frac{2}{2}$	5	-1
$B_{4,b}$		$2 + \frac{2}{4} + \frac{2}{7}$	$2 + \frac{1}{3} + \frac{1}{2}$	5	$\frac{1}{6}$ ⋆

Red edges do not belong to the block but indicate incidence.
Blue edges do not belong to the block and indicate an adjacent 4-face.

(iii) Proof for upper bound: Triangular-blocks

B	Diagram	$f(B) \leq$	$n(B) \leq$	$e(B) =$	$7f + 2n - 5e \leq$
$B_{4,b}$		$2 + \frac{2}{4} + \frac{2}{4}$	$2 + \frac{2}{3}$	5	$\frac{4}{3}$ ⋆
B_3		$1 + \frac{2}{7} + \frac{1}{4}$	$\frac{3}{2}$	3	$-\frac{5}{4}$
B_3		$1 + \frac{3}{4}$	$\frac{2}{2} + \frac{1}{3}$	3	$-\frac{1}{12}$
B_2		$\frac{1}{4} + \frac{1}{7}$	$\frac{2}{2}$	1	$-\frac{1}{4}$
B_2		$\frac{1}{4} + \frac{1}{7}$	$\frac{1}{2} + \frac{1}{3}$	1	$-\frac{7}{12}$
B_2		$\frac{1}{4} + \frac{1}{5}$	$\frac{2}{3}$	1	$-\frac{31}{60}$

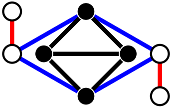
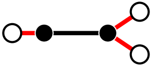
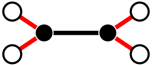
(iii) Proof for upper bound: Triangular-blocks

B	Diagram	$f(B) \leq$	$n(B) \leq$	$e(B) =$	$7f + 2n - 5e \leq$
$B_{5,d}$		$4 + \frac{2}{4} + \frac{2}{7}$	$2 + \frac{3}{2}$	8	$\frac{1}{2}$ ⋆
$B_{4,b}$		$2 + \frac{2}{4} + \frac{2}{7}$	$2 + \frac{1}{3} + \frac{1}{2}$	5	$\frac{1}{6}$ ⋆
B_2					$-\frac{1}{4}$

Blue edges:

- Form a K_2 triangular-block.
- Cannot be incident to two 4-faces.

(iii) Proof for upper bound: Triangular-blocks

B	Diagram	$f(B) \leq$	$n(B) \leq$	$e(B) =$	$7f + 2n - 5e \leq$
$B_{4,b}$		$2 + \frac{2}{4} + \frac{2}{4}$	$2 + \frac{2}{3}$	5	$\frac{4}{3}$ ⋆
B_2		$\frac{1}{4} + \frac{1}{7}$	$\frac{1}{2} + \frac{1}{3}$	1	$-\frac{7}{12}$
B_2		$\frac{1}{4} + \frac{1}{5}$	$\frac{2}{3}$	1	$-\frac{31}{60}$

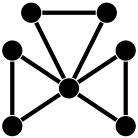
Blue edges:

- Form a K_2 triangular-block.
- Cannot be incident to two 4-faces.

Open problems

- Obtain bounds on $\mathrm{ex}_{\mathcal{P}}(n, C_k)$ for any $k \geq 7$.
- Obtain bounds for $\mathrm{ex}_{\mathcal{P}}(n, \{C_k, C_\ell\})$ for distinct $k, \ell \geq 3$
- [Lan-Shi-Song] If $n \geq 15$, then

$$\left\lfloor \frac{5n}{2} \right\rfloor \leq \mathrm{ex}_{\mathcal{P}}(n, K_1 \vee 2K_3) \leq \frac{17}{6}n - 4$$



$K_1 \vee 3K_2$

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