The edit distance on graphs, Part II

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5th Lake Michigan Workshop on Combinatorics and Graph Theory
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Survey: The edit distance in graphs: methods, results and generalizations, *Recent Trends in Combinatorics*, 31–62, IMA Vol. Math. Appl., **159**, Springer, Cham, 2016.

Structure

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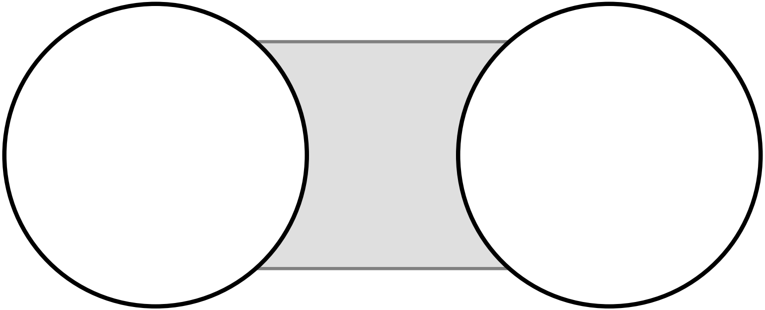
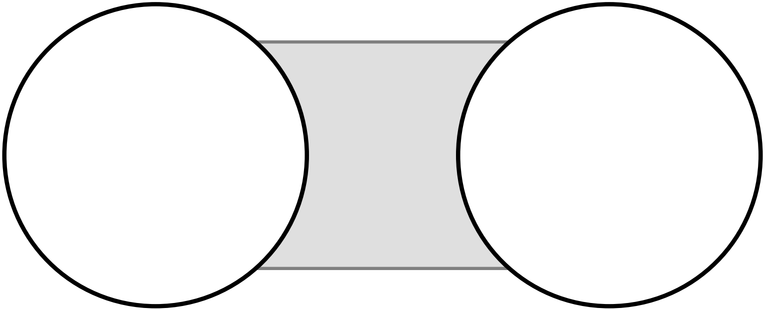
We can eliminate induced copies of C_5 by partitioning the vertices in half and deleting the edges in each part.



Structure

Let's revisit $\mathrm{Forb}(C_5)$.

We can eliminate induced copies of C_5 by partitioning the vertices in half *randomly* and deleting the edges in each part.



Structure

Let's revisit $\mathrm{Forb}(C_5)$.

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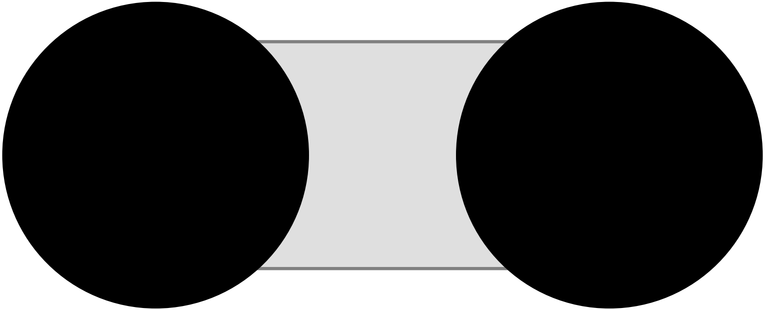

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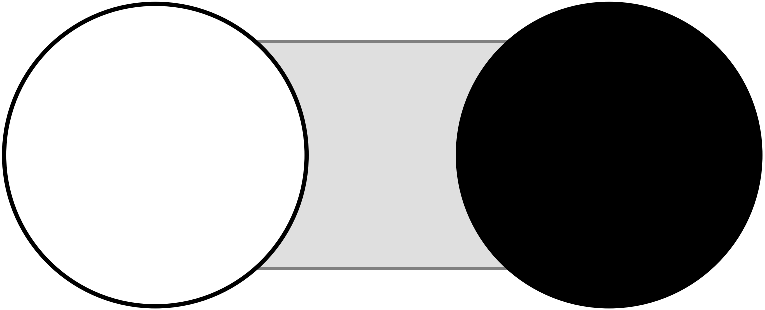

CRGs

Definition

A COLORED REGULARITY GRAPH (CRG), $K = (V, E)$, is a complete graph such that

- $V(K) = \mathrm{VW} \mathbin{\dot\cup} \mathrm{VB}$ (vertices are white and black)
- $E(K) = \mathrm{EW} \mathbin{\dot\cup} \mathrm{EG} \mathbin{\dot\cup} \mathrm{EB}$ (edges are white, gray, and black)

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Definition

A COLORED HOMOMORPHISM, $\varphi : V(H) \to V(K)$, is a map from a graph $H = (V, E)$ to a CRG $K = (\mathrm{VW}, \mathrm{VB}; \mathrm{EW}, \mathrm{EG}, \mathrm{EB})$ such that

- $vw \in E(\overline{H}) \implies \varphi(v) = \varphi(w) \in \mathrm{VW}$ or $\varphi(v)\varphi(w) \in \mathrm{EW} \cup \mathrm{EG}$.
- $vw \in E(H) \implies \varphi(v) = \varphi(w) \in \mathrm{VB}$ or $\varphi(v)\varphi(w) \in \mathrm{EB} \cup \mathrm{EG}$.

If there exists a colored homomorphism from H to K, then $H \mapsto K$.

When there is no map

If

- H is a graph,
- K is a CRG, and
- $H \not\mapsto K$, then

a graph G that "looks like K" has no induced copy of H.

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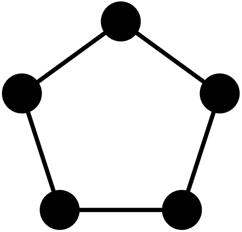
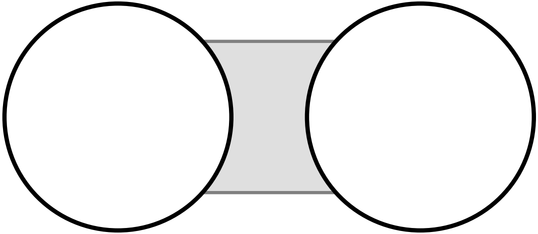


$G = C_5$



$K = K(2,0)$

A graph that is bipartite has no induced C_5.

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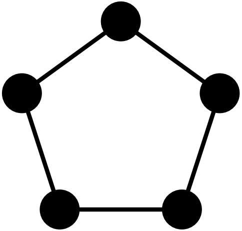

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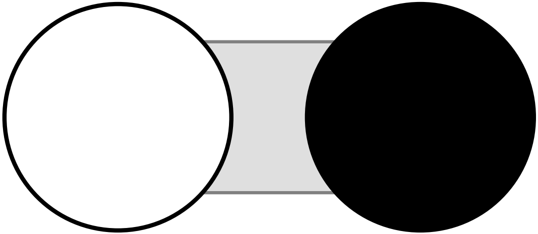

$K = K(1,1)$

A graph that is split has no induced C_5.

When there is no map

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- K is a CRG, and
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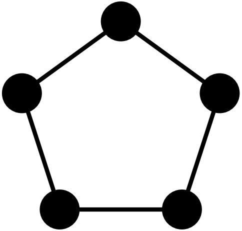

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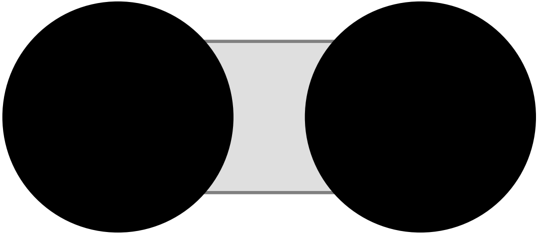

$K = K(0,2)$

A graph that is the complement of a bipartite graph has no induced C_5.

When there is a map

How to edit

Let G be a density-p graph. Let us transform it into a $G' \in \mathrm{Forb}(H)$.

1. Find a CRG $K = (\mathrm{VW}, \mathrm{VB}; \mathrm{EW}, \mathrm{EG}, \mathrm{EB})$ such that $H \not\mapsto K$.

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1. Find a CRG $K = (\mathrm{VW}, \mathrm{VB}; \mathrm{EW}, \mathrm{EG}, \mathrm{EB})$ such that $H \not\mapsto K$.
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3. If $\varphi(v) = \varphi(w) \in \mathrm{VW}$ or $\varphi(v)\varphi(w) \in \mathrm{EW}$ and $vw \in E(G)$, then delete vw.

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4. If $\varphi(v) = \varphi(w) \in \mathrm{VB}$ or $\varphi(v)\varphi(w) \in \mathrm{EB}$ and $vw \in E(\overline{G})$, then add vw.

How many changes were made, on average? $(k = |V(K)|)$

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How many changes were made, on average? $(k = |V(K)|)$

$$\mathbb{E}\left[\#\text{ of changes}\right] = \sum_{vw\in E(G)} \Pr\left\{\varphi(v)\varphi(w) \in (\mathrm{VW}\cup\mathrm{EW})\right\} + \sum_{vw\in E(\overline{G})} \Pr\left\{\varphi(v)\varphi(w) \in (\mathrm{VB}\cup\mathrm{EB})\right\}$$

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&= \sum_{vw \in E(G)} \frac{|\mathrm{VW}| + 2|\mathrm{EW}|}{k^2} + \sum_{vw \in E(\overline{G})} \frac{|\mathrm{VB}| + 2|\mathrm{EB}|}{k^2}
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&= \frac{|\mathrm{VW}| + 2|\mathrm{EW}|}{k^2}\, p\binom{n}{2} + \frac{|\mathrm{VB}| + 2|\mathrm{EB}|}{k^2}\,(1-p)\binom{n}{2}
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$$\mathrm{dist}(G, \mathrm{Forb}(H)) \leq \frac{1}{k^2}\left[p\left(|\mathrm{VW}| + 2|\mathrm{EW}|\right) + (1-p)\left(|\mathrm{VB}| + 2|\mathrm{EB}|\right)\right]$$

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The infimum is taken over all K such that $H \not\mapsto K$.
We denote

$$\mathcal{K}\left(\mathrm{Forb}(H)\right) := \left\{ K : H \not\mapsto K \right\}.$$

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Let G be a density-p graph. Let us transform it into a $G' \in \mathcal{H}$.
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$$\mathrm{dist}(G, \mathcal{H}) \leq \inf_K \left\{ \frac{1}{k^2} \left[p\left(|\mathrm{VW}| + 2|\mathrm{EW}|\right) + (1-p)\left(|\mathrm{VB}| + 2|\mathrm{EB}|\right) \right] \right\}$$

The infimum is taken over all K such that $H \not\mapsto K$, $\forall H \in \mathcal{F}(\mathcal{H})$.
We denote

$$\mathcal{K}(\mathcal{H}) := \{K : H \not\mapsto K, \forall H \in \mathcal{F}(\mathcal{H})\}.$$

where $\mathcal{H} = \bigcap \{\mathrm{Forb}(H) : H \in \mathcal{F}(\mathcal{H})\}$.

Best possible

For any CRG, $K = (\mathrm{VW}, \mathrm{VB}; \mathrm{EW}, \mathrm{EG}, \mathrm{EB})$, denote

$$f_K(p) = \frac{1}{k^2}\left[p\left(|\mathrm{VW}| + 2|\mathrm{EW}|\right) + (1-p)\left(|\mathrm{VB}| + 2|\mathrm{EB}|\right)\right].$$

If $\mathcal{K}(\mathcal{H}) = \{K : H \not\mapsto K\}$, then

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Consider the Erdős-Rényi random graph model $G(n,p)$.
The graph has n vertices and each pair has an edge with probability p.

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- Choose $G \sim G(n, p)$.
- Let $G' \in \mathcal{H}$ be the closest graph to G.

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- Apply Szemerédi's Regularity Lemma *to G'* with ϵ.

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- Let $G' \in \mathcal{H}$ be the closest graph to G.
- Apply Szemerédi's Regularity Lemma *to G'* with ϵ.
- We need to apply it twice – once to G' and once inside each cluster. A version of RegLem due to Alon, Fischer, Krivelevich, and M. Szegedy [2000] does this efficiently.

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- Choose $G \sim G(n, p)$.
- Let $G' \in \mathcal{H}$ be the closest graph to G.
- Apply Szemerédi's Regularity Lemma *to G'* with ϵ.
- Edit the very dense and very sparse regular pairs and clusters to look like $K \in \mathcal{K}(\mathcal{H})$.

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- Edit the very dense and very sparse regular pairs and clusters to look like $K \in \mathcal{K}(\mathcal{H})$.
- $\#$ of changes is $\geq f_K(p)\binom{n}{2} - \frac{\epsilon}{2}n^2$ with high probability.

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If $\mathcal{K}(\mathcal{H}) = \{K : H \not\mapsto K\}$, then

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- # of changes is $\geq f_K(p)\binom{n}{2} - \frac{\epsilon}{2}n^2$ with high probability.

For every $\epsilon > 0$, there exists a $\mathcal{K}(\mathcal{H}) = \{K : H \not\mapsto K\}$, such that with high probability,

$$\mathrm{dist}(G(n, p), \mathcal{H}) \geq f_K(p) - \epsilon.$$

The theorem revisited

Theorem (Balogh-M., 2008)

For every hereditary property, $\mathcal{H}$, and every $p \in [0,1]$,

$$\begin{aligned} \mathrm{ed}_{\mathcal{H}}(p) &= \lim_{n\to\infty} \mathbb{E}\left[\mathrm{dist}\left(G(n,p),\mathcal{H}\right)\right] \\ &= \lim_{n\to\infty} \max\left\{\mathrm{dist}\left(G,\mathcal{H}\right) : |V(G)| = n, |E(G)| = \left\lfloor p\binom{n}{2}\right\rfloor\right\} \end{aligned}$$

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So we only need to check all CRGs in $\mathcal{K}(\mathcal{H})$.

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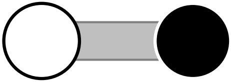
So we only need to check all CRGs in $\mathcal{K}(\mathcal{H})$. $|\mathcal{K}(\mathcal{H})| = \infty$.
But there are some important features of $K \in \mathcal{K}(\mathcal{H})$ we can exploit.

Weighting



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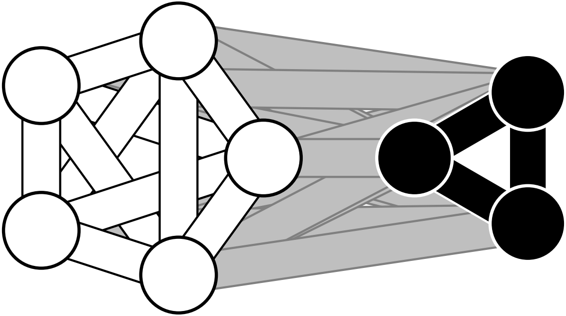

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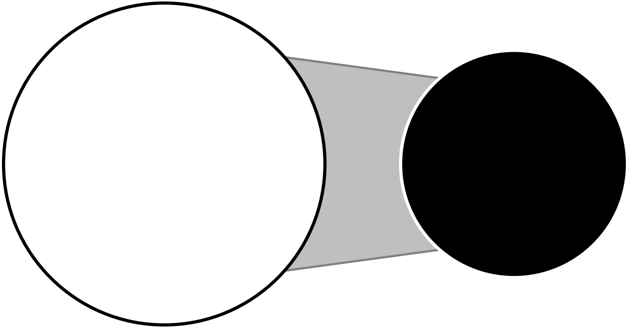

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where $\mathbf{1}^T = [1, 1, \ldots, 1]$ and

$$(\mathbf{M})_{ij} = \begin{cases} p, & \text{if } \{i,j\} \in \mathrm{VW} \cup \mathrm{EW}; \\ 1-p, & \text{if } \{i,j\} \in \mathrm{VB} \cup \mathrm{EB}; \\ 0, & \text{else.} \end{cases}$$

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p-cores

Theorem (Balogh-M., 2008)

For every hereditary property, $\mathcal{H}$, and every $p \in [0,1]$,

$$\begin{aligned}
\mathrm{ed}_{\mathcal{H}}(p) &= \lim_{n\to\infty} \mathbb{E}\left[\mathrm{dist}\left(G(n,p),\mathcal{H}\right)\right] \\
&= \lim_{n\to\infty} \max\left\{\mathrm{dist}\left(G,\mathcal{H}\right) : |V(G)| = n, |E(G)| = \left\lfloor p\binom{n}{2}\right\rfloor\right\} \\
&= \inf\left\{f_K(p) : K \in \mathcal{K}(\mathcal{H})\right\} \\
&= \inf\left\{g_K(p) : K \in \mathcal{K}(\mathcal{H})\right\}
\end{aligned}$$

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Definition

For $p \in (0,1)$, a CRG K is called p-CORE if

$$g_K(p) < g_{K'}(p)$$

for any proper sub-CRG K'.

Structure of p-cores

Consider a p-core CRG K and the following quadratic program:

$$g_K(p) = \min\left\{\mathbf{x}^T\mathbf{M}\mathbf{x} : \mathbf{x}^T\mathbf{1} = 1, \mathbf{x} \geq \mathbf{0}\right\}.$$

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Proposition

If the matrix $\mathbf{M}$ has three entries and the optimal $\mathbf{x}$ of the above quadratic program has no zero entries, then for $i \neq j$,

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This allows us to categorize p-core CRGs:

Theorem (Marchant-Thomason, 2010)

Let $p \in [0,1]$ and let $K = (\mathrm{VW}, \mathrm{VB}; \mathrm{EW}, \mathrm{EG}, \mathrm{EB})$ be a p-core CRG.

(a) $p \leq 1/2 \implies \mathrm{EB} = \emptyset$ *and no* EW *is incident to* VW.

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In particular, $1/2$-core CRGs have only gray edges. Hence the result

$$\mathrm{ed}_{\mathcal{H}}\left(\frac{1}{2}\right) = \frac{1}{2\left(\chi_{\mathrm{B}}(\mathcal{H}) - 1\right)}.$$

Gray degree

Consider a p-core CRG K and the following quadratic program:

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Theorem

For each vertex, v in a CRG K, with $g := g_K(p)$ as defined by the above quadratic program, we denote $\mathbf{x}(v)$ to be the weight of v in the optimal $\mathbf{x}$ and $d_G(v)$ the sum of the weights of the gray neighbors of v.

1. $p \leq 1/2 \implies \qquad d_G(v) = \qquad \dfrac{p-g}{p} + \dfrac{1-2p}{p}\mathbf{x}(v), \ \forall v \in \mathrm{VB}.$
2. $p \geq 1/2 \implies \qquad d_G(v) = \dfrac{1-p-g}{1-p} + \dfrac{2p-1}{1-p}\mathbf{x}(v), \ \forall v \in \mathrm{VW}.$

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Corollary

1. $p \leq 1/2 \implies \quad \mathbf{x}(v) \leq \dfrac{g}{1-p}, \ \forall v \in \mathrm{VB}.$
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Archipelago theorem

Definition

Let K be a CRG. A sub-CRG K' is a COMPONENT if, $\forall v_1, v_2 \in V(K')$, there exists a path in K' from v_1 to v_2 without using a gray edge.

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If K is a CRG with components $K_1, \ldots, K_\ell$, then

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If K has only gray edges, $r = |\mathrm{VW}|$, and $s = |\mathrm{VB}|$, then it is $K(r,s)$ and

$$g_{K(r,s)}(p) = \left(\frac{r}{p} + \frac{s}{1-p}\right)^{-1} = \frac{p(1-p)}{r(1-p)+sp}.$$

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An upper bound for $\mathrm{ed}_{\mathcal{H}}(p)$ comes from considering only gray-edge CRGs.

$$\begin{aligned} \mathrm{ed}_{\mathcal{H}}(p) &= \min\{g_K(p) : K \not\mapsto \mathcal{K}(\mathcal{H})\} \\ \gamma_{\mathcal{H}}(p) &:= \min\{g_K(p) : K \not\mapsto \mathcal{K}(\mathcal{H}), E(K) = \mathrm{EG}(K)\} \end{aligned}$$

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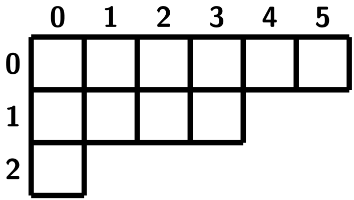
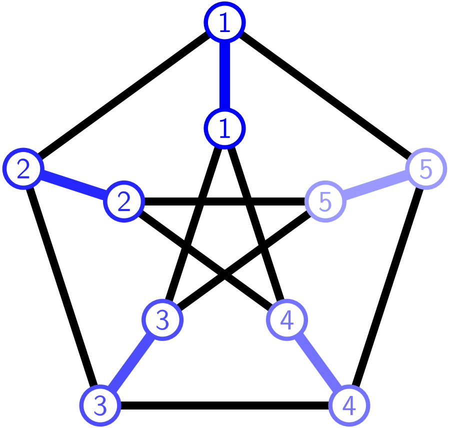
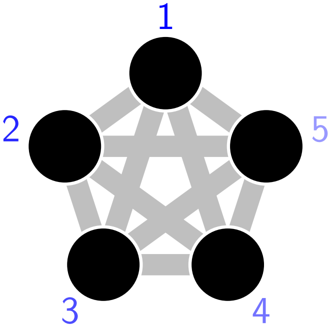
The set of (r,s) such that $K(r,s) \in \mathcal{K}(\mathcal{H})$ forms the CLIQUE SPECTRUM, a Young tableau.

$\Gamma(\mathrm{Forb}(C_{11}))$:



Let P_{10} be the Petersen graph.
Let $\Gamma(\mathrm{Forb}(P_{10}))$ be the clique spectrum of $\mathrm{Forb}(P_{10})$.





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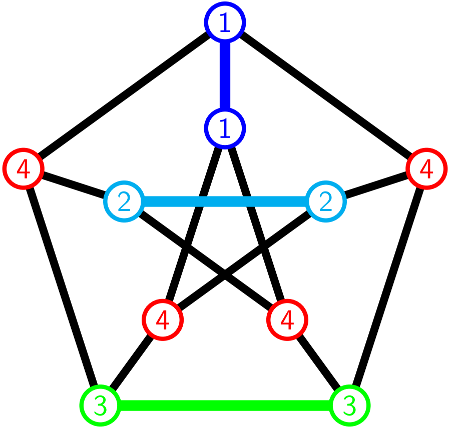

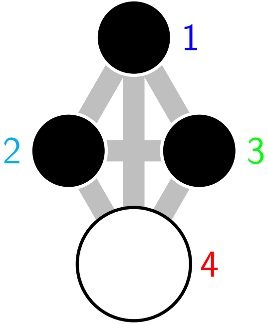

$$(1,3) \notin \Gamma(\mathrm{Forb}(P_{10}))$$

$(2,2) \notin \Gamma$

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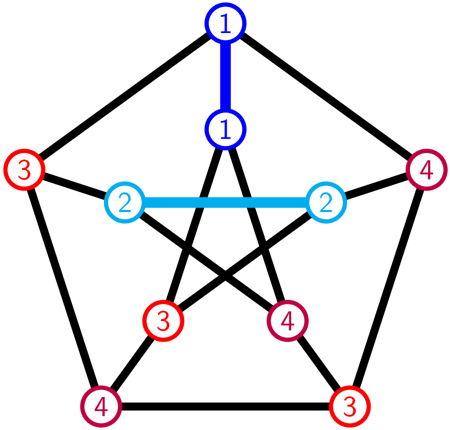

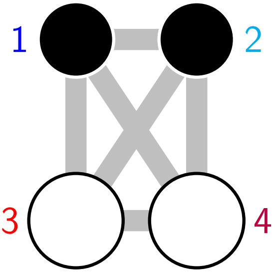

$$(2,2) \notin \Gamma(\mathrm{Forb}(P_{10}))$$

Clique spectrum

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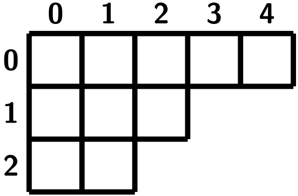

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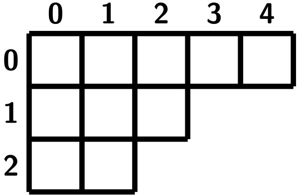


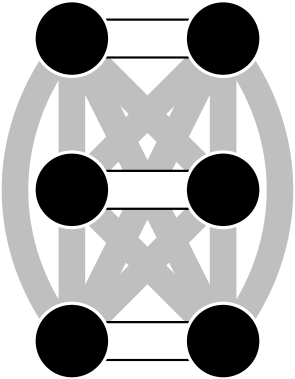
$$g_{K(2,1)}(p) = \left(\frac{2}{p} + \frac{1}{1-p}\right)^{-1} = \frac{p(1-p)}{2-p}$$

$$g_{K(1,2)}(p) = \left(\frac{1}{p} + \frac{2}{1-p}\right)^{-1} = \frac{p(1-p)}{1+p}$$

$$g_{K(0,4)}(p) = \left(\frac{0}{p} + \frac{4}{1-p}\right)^{-1} = \frac{1-p}{4}$$

$$\gamma_{\mathrm{Forb}(P_{10})} = \min\left\{\frac{p(1-p)}{2-p}, \frac{p(1-p)}{1+p}, \frac{1-p}{4}\right\}$$

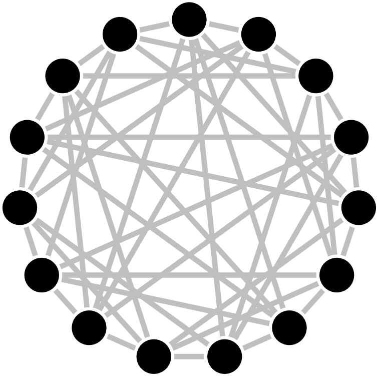
Simple gadgets



$$K_{2,5} \not\mapsto K$$

$$g_K(p) = \frac{1}{6}, \text{ if } p < 1/2.$$

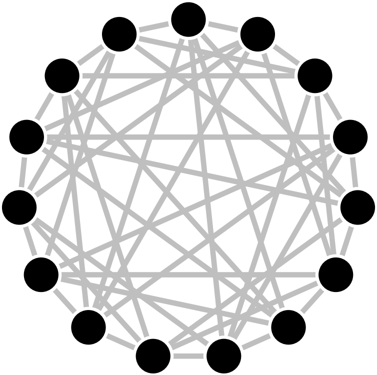
Strongly-regular graphs



$$K_{2,4} \not\mapsto K$$

$$g_K(p) = \frac{1+7p}{15}, \text{ if } p < 1/2.$$

Strongly-regular graphs



t	=	4
k	=	15
d	=	6

$$K_{2,t} \not\mapsto \mathrm{SRG}(k, d, \leq t-3, \leq t-1)$$

$$g_K(p) = \frac{1+(k-d-2)p}{k}, \text{ if } p < 1/2.$$

Zarankiewicz constructions

W.G. Brown constructed dense $K_{3,3}$-free graphs.

Lemma

If K is a CRG with $V(K) = \mathrm{VB}$ and $E(K) = \mathrm{EW} \,\dot\cup\, \mathrm{EG}$, then $K_{3,3} \not\mapsto K$ iff K does not have a gray K_3 or a gray $K_{3,3}$.

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- Given a (looped) graph G with no $K_{3,3}$, create a bipartite G' by cloning each vertex.
- For distinct $x, y \in V(G)$, let $x_1 \sim_{G'} y_2$ iff $x \sim_G y$.
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$$f_K(p) = \frac{1-2p}{2n} + p - \frac{pm}{n^2}.$$

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If $p < 1/124$, there is such a K for which
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$$f_K(p) = \frac{1-2p}{2n} + p - \frac{pm}{n^2}.$$

Theorem (Marchant-Thomason, 2010)

If $p < 1/124$, there is such a K for which

$$f_K(p) < \frac{p(1-p)}{1+p} = \gamma_{\mathrm{Forb}(K_{3,3})}(p).$$

Something similar works for $K_{2,t}$ (constructions due to Z. Füredi).

What's next?

We have most of the tools we need in order to compute an edit distance function.

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Part III:

- Step-by-step computation of a new edit distance function.
- Open problems and conjectures.