TAO'S SPECTRAL PROOF OF THE SZEMERÉDI REGULARITY LEMMA

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ABSTRACT. On December 3, 2012, following the Third Abel conference, in honor of Endre Szemerédi, Terence Tao posted on his blog a proof of the spectral version of Szemerédi's regularity lemma. This, in turn, proves the original version.

1. INTRODUCTION

Tao attributes this proof to Frieze and Kannan [1].

One thing to observe is that this is a statement on matrices and graphs are just a consequence.

2. BASIC MATRIX VERSION

Lemma 1 (Szemerédi's regularity lemma, matrix version)**.** *Let T be a self-adjoint $n \times n$ matrix such that $\operatorname{tr}(T^2) \leq n^2$. Let V be the set of n indices and let $\epsilon > 0$. Then there exists an $M \leq M(\epsilon)$ and*

- *a decomposition of T into three matrices, $T = T_1 + T_2 + T_3$, each of which is self-adjoint,*
- *a partition $V = V_0 \cup V_1 \cup \cdots \cup V_M$, and*
- *a set of pairs $\Sigma \subset \binom{\{0,\ldots,M\}}{2}$ (which contains all pairs with 0),*

such that

- *for all $i, j \in \{0, 1, \ldots, M\}$, there exists d_{ij} such that for all $a \in V_i$ and $b \in V_j$, we have $|(T_1)_{ab} - d_{ij}| < \epsilon$,*
- *for all $i, j \in \{0, 1, \ldots, M\} - \Sigma$, we have $\sum\limits_{a \in V_i} \sum\limits_{b \in V_j} |(T_2)_{ab}|^2 < \epsilon |V_i||V_j|$,*
- *for all $i, j \in \{0, 1, \ldots, M\} - \Sigma$, we have $n \cdot \sigma(T_3) < \epsilon |V_i||V_j|$, (where $\sigma(T_3)$ is T_3's largest singular value) and*
- $\sum\limits_{(i,j) \in \Sigma} |V_i||V_j| \leq \epsilon n^2$.

Proof. Enumerate $V = \{1, \ldots, n\}$. Since T is self-adjoint, it has an eigenvalue decomposition

$$T = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{u}_i^*,$$

for some orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of $\mathbb{C}^n$ (where the vectors are column vectors) and real eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. We arrange them in decreasing order of magnitude

$$|\lambda_1| \geq \cdots \geq |\lambda_n|.$$

In a self-adjoint matrix, the trace of T^2 is the sum of the squares of the eigenvalues of T and so $\mathrm{tr}(T^2) = \sum_{j=1}^n |\lambda_j|^2$. So we can bound the eigenvalues by observing that $i\lambda_i^2 \le \sum_{j=1}^i \lambda_j^2 \le n^2$ and so, for all $i \in \{1, \dots, n\}$,

$$|\lambda_i| \le \frac{n}{\sqrt{i}}. \tag{1}$$

We will be given a function $F : \mathbb{N} \to \mathbb{N}$ that we will specify later. This function does depend on ϵ (we suppress this in the notation) and it satisfies the inequality $F(i) > i$ for all integers i. We want to find an integer J for which

$$\sum_{J \le j < F(J)} |\lambda_j|^2 \le \epsilon^3 n^2. \tag{2}$$

To do this, we consider the partition of $\{1, \dots, n\}$ into intervals $[F^{(k-1)}(1), F^{(k)}(1) - 1]$ from $k = 1, \dots, 1/\epsilon^3$ where $F^{(k)}$ represents the k^{th} composition of F with itself. Note that either we find a $J = F^{(k-1)}(1)$ for which (2) is satisfied or the sum of $|\lambda_j|^2$ for all j in some interval is greater than $\epsilon^3 n^2$. Since there are $1/\epsilon^3$ intervals, this would contradict the n^2 bound for $\mathrm{tr}(T^2)$.

Thus, we have a partition of T into three matrices:

$$T = T_1 + T_2 + T_3,$$

where T_1 is the "structured" component

$$T_1 := \sum_{i<J} \lambda_i \mathbf{u}_i \mathbf{u}_i^*, \tag{3}$$

and T_2 is the "error" component

$$T_2 := \sum_{J \le i < F(J)} \lambda_i \mathbf{u}_i \mathbf{u}_i^*, \tag{4}$$

and T_3 is the "pseudorandom" component

$$T_3 := \sum_{i \ge F(J)} \lambda_i \mathbf{u}_i \mathbf{u}_i^*. \tag{5}$$

We will partition the vertex set so that T_1 is approximately constant on most clusters. The number of such clusters will be $O_{J,\epsilon}(1)$. For each $j < J$ we define a partition into clusters on which entry $\mathbf{u}_i$ (a complex number) varies by $\frac{\epsilon}{J} n^{-1/2}$. There is also an exceptional cluster of size $\frac{\epsilon}{J} n$ which comes from the vertices for which the entry of $\mathbf{u}_i$ is large in magnitude. That is, either its real or its imaginary part is larger than $\sqrt{\frac{J}{\epsilon}}\, n^{-1/2}$ in absolute value.

To see this, simply place a vertex into the exceptional cluster if the corresponding entry of $\mathbf{u}_i$ has the absolute value of either its real or imaginary part at least $\sqrt{\frac{J}{\epsilon}}\, n^{-1/2}$. Since $\|\mathbf{u}_i\|_2 = 1$, this means there can be at most $\frac{\epsilon}{J} n$ such entries. Partition the square of length $2\sqrt{\frac{J}{\epsilon}} n^{-1/2}$ centered at the origin of the complex plane into subsquares of side length $\frac{\epsilon^{3/2}}{J^{3/2}} n^{-1/2}$. There are $\left(2\sqrt{\frac{J}{\epsilon}}\, n^{-1/2}\right)^2 / \left(\frac{\epsilon^{3/2}}{J^{3/2}}\, n^{-1/2}\right)^2 = 4J^4/\epsilon^4$ such subsquares. Partition the vertices according to where its corresponding entry of $\mathbf{u}_i$ lies.

Take the union of all vertices in the exceptional clusters and the corresponding exceptional cluster is of size at most $(J-1)\cdot\frac{\epsilon n}{J} < \epsilon n$. For the rest of the vertices, we take the common refinement of the partitions defined by each $\mathbf{u}_i$, $i < J$. This defines a partition of the vertex set $V = V_0 + V_1 + \cdots + V_M$ in which V_0 is the exceptional set. For $i = 1, \ldots, M$, the entries over V_i of each of $\mathbf{u}_1, \ldots, \mathbf{u}_{J-1}$ have magnitude at most

$$\sqrt{2}\cdot 2\sqrt{\frac{J}{\epsilon}}\, n^{-1/2}, \tag{6}$$

and differ in magnitude by at most

$$\sqrt{2}\,\frac{\epsilon^{3/2}}{J^{3/2}}\, n^{-1/2}. \tag{7}$$

Moreover,

$$M \le \left(\frac{4J^4}{\epsilon^4}\right)^J. \tag{8}$$

Now, let $i, j \in \{1, \ldots, M\}$. We will show that the values of T_1 over the block $V_i \times V_j$ differ by at most 8ϵ. To see this, let $a, c \in V_i$ and $b, d \in V_j$. Then

$$\begin{aligned}
(T_1)_{ab} - (T_1)_{cd} &= \left|\sum_{i<J} \lambda_i \mathbf{u}_i(a)\mathbf{u}_i(b) - \lambda_i \mathbf{u}_i(c)\mathbf{u}_i(d)\right| \\
&\le \sum_{i<J} |\lambda_i||\mathbf{u}_i(a)\mathbf{u}_i(b) - \mathbf{u}_i(c)\mathbf{u}_i(b) + \mathbf{u}_i(c)\mathbf{u}_i(b) - \mathbf{u}_i(c)\mathbf{u}_i(d)| \\
&\le \sum_{i<J} n|\mathbf{u}_i(b)||\mathbf{u}_i(a) - \mathbf{u}_i(c)| + n|\mathbf{u}_i(c)||\mathbf{u}_i(b) - \mathbf{u}_i(d)| \\
&\le J\left(n\cdot 2\sqrt{2}\sqrt{\frac{J}{\epsilon}}\, n^{-1/2}\cdot\sqrt{2}\frac{\epsilon^{3/2}}{J^{3/2}}\, n^{-1/2} + n\cdot 2\sqrt{2}\sqrt{\frac{J}{\epsilon}}\, n^{-1/2}\cdot\sqrt{2}\frac{\epsilon^{3/2}}{J^{3/2}}\, n^{-1/2}\right) \\
&= 2J\left(n\cdot 2\sqrt{2}\sqrt{\frac{J}{\epsilon}}\, n^{-1/2}\cdot\sqrt{2}\frac{\epsilon^{3/2}}{J^{3/2}}\, n^{-1/2}\right) \\
&< 8\epsilon.
\end{aligned}$$

As a result, we can conclude that, if d_{ij} is the mean of the entries in the block $V_i \times V_j$, then by the triangle inequality,

$$|(T_1)_{ab} - d_{ij}| < 16\epsilon. \tag{9}$$

Next we consider T_2 and observe that $\mathrm{tr}(T_2^2) = \sum_{J\le j\le F(J)} \lambda_i^2 < \epsilon^3 n^2$. So, $\sum_{a,b\in V} |(T_2)_{ab}|^2 < \epsilon^3 n^2$. Define Σ_1 so that for every $(i,j) \notin \Sigma_1$,

$$\sum_{a\in V_i}\sum_{b\in V_j} |(T_2)_{ab}|^2 < \epsilon|V_i||V_j|. \tag{10}$$

Thus,

$$\epsilon^2 \sum_{(i,j)\in\Sigma_1} |V_i||V_j| \le \sum_{(i,j)\in\Sigma_1}\sum_{a\in V_i}\sum_{b\in V_j} |(T_2)_{ab}|^2 \le \epsilon^3 n^2.$$

consequently,

$$\sum_{(i,j)\in\Sigma_1} |V_i||V_j| \le \epsilon n^2.$$

Finally, we turn our attention to T_3. The maximum eigenvalue of T_3 is $|\lambda_{F(J)}| \le n/\sqrt{F(J)}$. We want to establish that $n^2/\sqrt{F(J)} \le \epsilon |V_i||V_j|$ for $(i,j) \notin \Sigma$. Because $|V_i|, |V_j| \ge \epsilon n/M$, it is sufficient to show that $F(J) \ge M^4/\epsilon^6$ because that would verify that

$$\frac{n^2}{\sqrt{F(J)}} \le \frac{\epsilon^3 n^2}{M^2} \le \epsilon |V_i||V_j|.$$

By (8), $M \le (4J^4/\epsilon^4)^J$. So the function that suffices is

$$F(x) \ge \frac{1}{\epsilon^6}\left(\frac{4x^4}{\epsilon^4}\right)^{4x}.$$

Let Σ be the pairs $(i,j) \in \{0,1,\ldots,M\}$ such that either $(i,j) \in \Sigma_1$, $i = 0$, $j = 0$ or $\min(|V_i|,|V_j|) \le \frac{\epsilon n}{M}$. Thus,

$$\sum_{(i,j)\in\Sigma} |V_i||V_j| \le \sum_{(i,j)\in\Sigma_1} |V_i||V_j| + 2|V_0||V| + 2\sum_{|V_i|<\epsilon n/M} |V_i||V|$$

$$\le \epsilon n^2 + 2\epsilon n \cdot n + 2M\frac{\epsilon n}{M} n \le 5\epsilon n^2. \tag{11}$$

Note that in this proof we use coefficients of $1q6\epsilon$ and 5ϵ in (9) and (11), respectively. We can, of course, choose $\epsilon/16$ rather than ϵ but we chose these parameters to make the computations somewhat more transparent. □

3. Spectral version

Lemma 2 (Szemerédi's regularity lemma, spectral version). *Let T be a self-adjoint $n \times n$ matrix such that $\mathrm{tr}(T^2) \le n^2$. Let V be the set of n indices and let $\epsilon > 0$. Then there exists a partition $V = V_1 \cup \cdots \cup V_M$ for some $M \le M(\epsilon)$ with the property that, for all pairs $(i,j) \in \{1,\ldots,M\}^2$ outside of an exceptional set Σ, one has*

$$|\mathbf{v}_B^*(T - d_{ij}\mathbf{I})\mathbf{v}_A| \le \epsilon |V_i||V_j| \tag{12}$$

whenever $\mathrm{supp}(\mathbf{v}_A) \subset V_i$*,* $\|\mathbf{v}_A\|_2^2 \le |V_i|$, $\mathrm{supp}(\mathbf{v}_B) \subset V_j$ *and* $\|\mathbf{v}_B\|_2^2 \le |V_j|$*, for some real number d_{ij}. Furthermore, we have*

$$\sum_{(i,j)\in\Sigma} |V_i||V_j| \le \epsilon n^2. \tag{13}$$

Proof. Enumerate $V = \{1,\ldots,n\}$. Since T is self-adjoint, it has an eigenvalue decomposition

$$T = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{u}_i^*,$$

for some orthonormal basis $\mathbf{u}_1,\ldots,\mathbf{u}_n$ of $\mathbb{C}^n$ (where the vectors are column vectors) and real eigenvalues $\lambda_1,\ldots,\lambda_n \in \mathbb{R}$. We arrange them in decreasing order of magnitude

$$|\lambda_1| \ge \cdots \ge |\lambda_n|.$$

In a self-adjoint matrix, the trace of T^2 is the sum of the squares of the eigenvalues of T and so $\mathrm{tr}(T^2) = \sum_{j=1}^n |\lambda_j|^2$. So we can bound the eigenvalues by observing that $i\lambda_i^2 \le \sum_{j=1}^i \lambda_j^2 \le n^2$ and so, for all $i \in \{1, \dots, n\}$,

$$|\lambda_i| \le \frac{n}{\sqrt{i}}. \tag{14}$$

We will be given a function $F : \mathbb{N} \to \mathbb{N}$ that we will specify later. This function does depend on ϵ (we suppress this in the notation) and it satisfies the inequality $F(i) > i$ for all integers i. We want to find an integer J for which

$$\sum_{J \le j < F(J)} |\lambda_j|^2 \le \epsilon^3 n^2. \tag{15}$$

To do this, we consider the partition of $\{1, \dots, n\}$ into intervals $[F^{(k-1)}(1), F^{(k)}(1) - 1]$ from $k = 1, \dots, 1/\epsilon^3$ where $F^{(k)}$ represents the k^{th} composition of F with itself. Note that either we find a $J = F^{(k-1)}(1)$ for which (15) is satisfied or the sum of $|\lambda_j|^2$ for all j in some interval is greater than $\epsilon^3 n^2$. Since there are $1/\epsilon^3$ intervals, this would contradict the n^2 bound for $\mathrm{tr}(T^2)$.

Thus, we have a partition of T into three matrices:

$$T = T_1 + T_2 + T_3,$$

where T_1 is the "structured" component

$$T_1 := \sum_{i<J} \lambda_i \mathbf{u}_i \mathbf{u}_i^*, \tag{16}$$

and T_2 is the "error" component

$$T_2 := \sum_{J \le i < F(J)} \lambda_i \mathbf{u}_i \mathbf{u}_i^*, \tag{17}$$

and T_3 is the "pseudorandom" component

$$T_3 := \sum_{i \ge F(J)} \lambda_i \mathbf{u}_i \mathbf{u}_i^*. \tag{18}$$

We will partition the vertex set so that T_1 is approximately constant on most clusters. The number of such clusters will be $O_{J,\epsilon}(1)$. For each $j < J$ we define a partition into clusters on which entry $\mathbf{u}_i$ (a complex number) varies by $\frac{\epsilon}{J} n^{-1/2}$. There is also an exceptional cluster of size $\frac{\epsilon}{J} n$ which comes from the vertices for which the entry of $\mathbf{u}_i$ is large in magnitude. That is, either its real or its imaginary part is larger than $\sqrt{\frac{J}{\epsilon}}\, n^{-1/2}$ in absolute value.

To see this, simply place a vertex into the exceptional cluster if the corresponding entry of $\mathbf{u}_i$ has the absolute value of either its real or imaginary part at least $\sqrt{\frac{J}{\epsilon}}\, n^{-1/2}$. Since $\|\mathbf{u}_i\|_2 = 1$, this means there can be at most $\frac{\epsilon}{J} n$ such entries. Partition the square of length $2\sqrt{\frac{J}{\epsilon}} n^{-1/2}$ centered at the origin of the complex plane into subsquares of side length $\frac{\epsilon^{3/2}}{J^{3/2}}\, n^{-1/2}$. There are $\left(2\sqrt{\frac{J}{\epsilon}}\, n^{-1/2}\right)^2 / \left(\frac{\epsilon^{3/2}}{J^{3/2}}\, n^{-1/2}\right)^2 = 4J^4/\epsilon^4$ such subsquares. Partition the vertices according to where its corresponding entry of $\mathbf{u}_i$ lies.

Take the union of all vertices in the exceptional clusters and the corresponding exceptional cluster is of size at most $(J-1)\cdot\frac{\epsilon n}{J} < \epsilon n$. For the rest of the vertices, we take the common refinement of the partitions defined by each $\mathbf{u}_i$, $i < J$. This defines a partition of the vertex set $V = V_0 + V_1 + \cdots + V_M$ in which V_0 is the exceptional set. For $i = 1, \ldots, M$, the entries over V_i of each of $\mathbf{u}_1, \ldots, \mathbf{u}_{J-1}$ have magnitude at most

$$\sqrt{2}\cdot 2\sqrt{\frac{J}{\epsilon}}\,n^{-1/2}, \tag{19}$$

and differ in magnitude by at most

$$\sqrt{2}\,\frac{\epsilon^{3/2}}{J^{3/2}}\,n^{-1/2}. \tag{20}$$

Moreover,

$$M \leq \left(\frac{4J^4}{\epsilon^4}\right)^J. \tag{21}$$

Now, let $i, j \in \{1, \ldots, M\}$. We will show that the values of T_1 over the block $V_i \times V_j$ differ by at most 8ϵ. To see this, let $a, c \in V_i$ and $b, d \in V_j$. Then

$$\begin{aligned}
(T_1)_{ab} - (T_1)_{cd} &= \left|\sum_{i<J} \lambda_i \mathbf{u}_i(a)\mathbf{u}_i(b) - \lambda_i \mathbf{u}_i(c)\mathbf{u}_i(d)\right| \\
&\leq \sum_{i<J} |\lambda_i||\mathbf{u}_i(a)\mathbf{u}_i(b) - \mathbf{u}_i(c)\mathbf{u}_i(b) + \mathbf{u}_i(c)\mathbf{u}_i(b) - \mathbf{u}_i(c)\mathbf{u}_i(d)| \\
&\leq \sum_{i<J} n|\mathbf{u}_i(b)||\mathbf{u}_i(a) - \mathbf{u}_i(c)| + n|\mathbf{u}_i(c)||\mathbf{u}_i(b) - \mathbf{u}_i(d)| \\
&\leq J\left(n\cdot 2\sqrt{2}\sqrt{\frac{J}{\epsilon}}\,n^{-1/2}\cdot\sqrt{2}\frac{\epsilon^{3/2}}{J^{3/2}}\,n^{-1/2} + n\cdot 2\sqrt{2}\sqrt{\frac{J}{\epsilon}}\,n^{-1/2}\cdot\sqrt{2}\frac{\epsilon^{3/2}}{J^{3/2}}\,n^{-1/2}\right) \\
&= 2J\left(n\cdot 2\sqrt{2}\sqrt{\frac{J}{\epsilon}}\,n^{-1/2}\cdot\sqrt{2}\frac{\epsilon^{3/2}}{J^{3/2}}\,n^{-1/2}\right) \\
&< 8\epsilon.
\end{aligned}$$

As a result, we can conclude that, if d_{ij} is the mean of the entries in the block $V_i \times V_j$, then by the triangle inequality and Cauchy-Schwarz,

$$\begin{aligned}
|\mathbf{v}_B^*(T_1 - d_{ij}\mathbf{I})\mathbf{v}_A| &\leq \sum_{a\in V_i}\sum_{b\in V_j} |(T_1)_{ab} - d_{ij})|\,|\mathbf{v}_A(a)||\mathbf{v}_B(b)| \\
&< 8\epsilon\|\mathbf{v}_A\|_1\|\mathbf{v}_B\|_1 \\
&\leq 8\epsilon|V_i||V_j|.
\end{aligned}$$

The last step follows from a basic vector norm inequality which gives $\|\mathbf{v}_A\|_1 \leq \sqrt{|V_i|}\|\mathbf{v}_A\|_2 \leq |V_i|$ and $\|\mathbf{v}_B\|_1 \leq \sqrt{|V_j|}\|\mathbf{v}_B\|_2 \leq |V_j|$.

Next we consider T_2 and observe that $\mathrm{tr}(T_2^2) = \sum_{J\leq j\leq F(J)} \lambda_i^2 \leq \epsilon^3 n^2$.

So, $\sum_{a,b\in V} |(T_2)_{ab}|^2 \le \epsilon^3 n^2$. Define Σ_1 so that for every $(i,j) \notin \Sigma_1$,

$$\sum_{a\in V_i}\sum_{b\in V_j} |(T_2)_{ab}|^2 \le \epsilon |V_i||V_j|. \tag{22}$$

Thus,

$$\epsilon^2 \sum_{(i,j)\in\Sigma_1} |V_i||V_j| \le \sum_{(i,j)\in\Sigma_1}\sum_{a\in V_i}\sum_{b\in V_j} |(T_2)_{ab}|^2 \le \epsilon^3 n^2.$$

consequently,

$$\sum_{(i,j)\in\Sigma_1} |V_i||V_j| \le \epsilon n^2.$$

So, for any $(i,j) \notin \Sigma_1$, use the fact that $\sum_{a\in V_i}\sum_{b\in V_j} |(T_2)_{ab}|^2 \le \epsilon|V_i||V_j|$ we use Cauchy-Schwarz to obtain the following bound

$$\begin{aligned}
|\mathbf{v}_B^* T_2 \mathbf{v}_A|^2 &= \left|\sum_{a\in V_i}\sum_{b\in V_j} (T_2)_{ab}\mathbf{v}_A(a)\cdot\overline{\mathbf{v}_B(b)}\right|^2 \\
&\le \left(\sum_{a\in V_i}\sum_{b\in V_j} |(T_2)_{ab}|^2\right)\left(\sum_{a\in V_i}\sum_{b\in V_j} |\mathbf{v}_A(a)|^2 |\mathbf{v}_B(b)|^2\right) \\
&= \left(\epsilon^2|V_i||V_j|\right)\|\mathbf{v}_A\|_2^2\|\mathbf{v}_B\|_2^2 \\
|\mathbf{v}_B^* T_2 \mathbf{v}_A| &\le \epsilon|V_i||V_j|.
\end{aligned}$$

The last step follows from $\|\mathbf{v}_A\|_2^2 \le |V_i|$ and $\|\mathbf{v}_B\|_2^2 \le |V_j|$.

Finally, we turn our attention to T_3. Let $\mathbf{v}_A$ and $\mathbf{v}_B$ be vectors such that $\|\mathbf{v}_A\|_2^2, \|\mathbf{v}_B\|_2^2 \le n$. Since the maximum eigenvalue of T_3 is $|\lambda_{F(J)}| \le n/\sqrt{F(J)}$, we have, first by Cauchy-Schwarz,

$$|\mathbf{v}_B^* T_3 \mathbf{v}_A| \le |\lambda_{F(J)}|\|\mathbf{v}_A\|_2\|\mathbf{v}_B\|_2 \le n^2/\sqrt{F(J)}.$$

Let Σ be the pairs $(i,j) \in \{0,1,\ldots,M\}$ such that either $(i,j)\in\Sigma_1$, $i=0$, $j=0$ or $\min(|V_i|,|V_j|) \le \frac{\epsilon n}{M}$. Thus,

$$\begin{aligned}
\sum_{(i,j)\in\Sigma} |V_i||V_j| &\le \sum_{(i,j)\in\Sigma_1} |V_i||V_j| + 2|V_0||V| + 2\sum_{|V_i|<\epsilon n/M} |V_i||V| \\
&\le \epsilon n^2 + 2\epsilon n\cdot n + 2M\frac{\epsilon n}{M}n \le 5\epsilon n^2. \qquad (23)
\end{aligned}$$

If $(i,j)\notin\Sigma$, then for all $A\subseteq V_i$ and $B\subseteq V_j$,

$$\begin{aligned}
|\mathbf{v}_B^*(T-d_{ij})\mathbf{v}_A| &\le |\mathbf{v}_B^*(T_1-d_{ij})\mathbf{v}_A| + |\mathbf{v}_B^* T_2\mathbf{v}_A| + |\mathbf{v}_B^* T_3\mathbf{v}_A| \\
&\le 8\epsilon|V_i||V_j| + \epsilon|V_i||V_j| + n^2/\sqrt{F(J)}.
\end{aligned}$$

Finally, we want to establish that $n^2/\sqrt{F(J)} \le \epsilon|V_i||V_j|$ for $(i,j)\notin\Sigma$. This would establish that

$$|\mathbf{v}_B^*(T-d_{ij})\mathbf{v}_A| \le 10\epsilon|V_i||V_j|. \tag{24}$$

Because $|V_i|, |V_j| \geq \epsilon n/M$, it is sufficient to show that $F(J) \geq M^4/\epsilon^6$ because that would verify that

$$\frac{n^2}{\sqrt{F(J)}} \leq \frac{\epsilon^3 n^2}{M^2} \leq \epsilon |V_i||V_j|.$$

By (21), $M \leq (4J^4/\epsilon^4)^J$. So the function that suffices is

$$F(x) \geq \frac{1}{\epsilon^6}\left(\frac{4x^4}{\epsilon^4}\right)^{4x}.$$

Note that in this proof we use coefficients of 10ϵ and 5ϵ in (24) and (23), respectively. We can, of course, choose $\epsilon/10$ rather than ϵ but we chose these parameters to make the computations somewhat more transparent. □

4. Graph version

Lemma 3 (Szemerédi's regularity lemma, spectral version)**.** *Let $G = (V, E)$ be a graph on n vertices and let $\epsilon > 0$. Then there exists a partition $V = V_1 \cup \cdots \cup V_M$ for some $M \leq M(\epsilon)$ with the property that, for all pairs $(i, j) \in \{1, \ldots, M\}^2$ outside of an exceptional set Σ, one has*

$$|E(A, B) - d_{ij}|A||B|| \ll \epsilon |V_i||V_j| \tag{25}$$

whenever $A \subset V_i$, $B \subset V_j$, for some real number d_{ij}, where

$$E(A, B) := |\{(a, b) \in A \times B : \{a, b\} \in E\}|$$

is the number of edges between A and B. Furthermore, we have

$$\sum_{(i,j)\in\Sigma} |V_i||V_j| \ll \epsilon |V|^2. \tag{26}$$

Proof. Here we do the proof directly, even though this is a direct consequence of the matrix version above.

Enumerate $V = \{1, \ldots, n\}$. Let T be the incidence matrix of G and note that since T is self-adjoint, it has an eigenvalue decomposition

$$T = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{u}_i^*,$$

for some orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of $\mathbb{C}^n$ (where the vectors are column vectors) and real eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. We arrange them in decreasing order of magnitude

$$|\lambda_1| \geq \cdots \geq |\lambda_n|.$$

In a self-adjoint matrix, the trace of T^2 is the sum of the squares of the eigenvalues of T and so $\operatorname{tr}(T^2) = \sum_{j=1}^{n} |\lambda_j|^2$. In addition, it is the sum of the degrees of degrees of the graph, hence $\operatorname{tr}(T^2) = 2|E(G)| \leq n^2$. So we can bound the eigenvalues by observing that $i\lambda_i^2 \leq \sum_{j=1}^{i} \lambda_j^2 \leq n^2$ and so, for all $i \in \{1, \ldots, n\}$,

$$|\lambda_i| \leq \frac{n}{\sqrt{i}}. \tag{27}$$

We will be given a function $F : \mathbb{N} \to \mathbb{N}$ that we will specify later. This function does depend on ϵ (we suppress this in the notation) and it satisfies the inequality $F(i) > i$ for all integers i. We want to find an integer J for which

$$\sum_{J \le j < F(J)} |\lambda_j|^2 \le \epsilon^3 n^2. \tag{28}$$

To do this, we consider the partition of $\{1, \dots, n\}$ into intervals $[F^{(k-1)}(1), F^{(k)}(1) - 1]$ from $k = 1, \dots, 1/\epsilon^3$ where $F^{(k)}$ represents the k^{th} composition of F with itself. Note that either we find a $J = F^{(k-1)}(1)$ for which (28) is satisfied or the sum of $|\lambda_j|^2$ for all j in some interval is greater than $\epsilon^3 n^2$. Since there are $1/\epsilon^3$ intervals, this would contradict the n^2 bound for $\mathrm{tr}(T^2)$.

Thus, we have a partition of T into three matrices:

$$T = T_1 + T_2 + T_3,$$

where T_1 is the "structured" component

$$T_1 := \sum_{i<J} \lambda_i \mathbf{u}_i \mathbf{u}_i^*, \tag{29}$$

and T_2 is the "error" component

$$T_2 := \sum_{J \le i < F(J)} \lambda_i \mathbf{u}_i \mathbf{u}_i^*, \tag{30}$$

and T_3 is the "pseudorandom" component

$$T_3 := \sum_{i \ge F(J)} \lambda_i \mathbf{u}_i \mathbf{u}_i^*. \tag{31}$$

We will partition the vertex set so that T_1 is approximately constant on most clusters. The number of such clusters will be $O_{J,\epsilon}(1)$. For each $j < J$ we define a partition into clusters on which entry $\mathbf{u}_i$ (a complex number) varies by $\frac{\epsilon}{J} n^{-1/2}$. There is also an exceptional cluster of size $\frac{\epsilon}{J} n$ which comes from the vertices for which the entry of $\mathbf{u}_i$ is large in magnitude. That is, either its real or its imaginary part is larger than $\sqrt{\frac{J}{\epsilon}}\, n^{-1/2}$ in absolute value.

To see this, simply place a vertex into the exceptional cluster if the corresponding entry of $\mathbf{u}_i$ has the absolute value of either its real or imaginary part at least $\sqrt{\frac{J}{\epsilon}}\, n^{-1/2}$. Since $\|\mathbf{u}_i\|_2 = 1$, this means there can be at most $\frac{\epsilon}{J} n$ such entries. Partition the square of length $2\sqrt{\frac{J}{\epsilon}} n^{-1/2}$ centered at the origin of the complex plane into subsquares of side length $\frac{\epsilon^{3/2}}{J^{3/2}}\, n^{-1/2}$. There are $\left(2\sqrt{\frac{J}{\epsilon}}\, n^{-1/2}\right)^2 / \left(\frac{\epsilon^{3/2}}{J^{3/2}}\, n^{-1/2}\right)^2 = 4J^4/\epsilon^4$ such subsquares. Partition the vertices according to where its corresponding entry of $\mathbf{u}_i$ lies.

Take the union of all vertices in the exceptional clusters and the corresponding exceptional cluster is of size at most $(J-1) \cdot \frac{\epsilon n}{J} < \epsilon n$. For the rest of the vertices, we take the common refinement of the partitions defined by each $\mathbf{u}_i$, $i < J$. This defines a partition of the vertex set $V = V_0 + V_1 + \cdots + V_M$ in which V_0 is the exceptional set. For $i = 1, \dots, M$, the entries over V_i of each of $\mathbf{u}_1, \ldots, \mathbf{u}_{J-1}$ have magnitude at most

$$\sqrt{2} \cdot 2\sqrt{\frac{J}{\epsilon}}\, n^{-1/2}. \tag{32}$$

and differ in magnitude by at most

$$\sqrt{2}\, \frac{\epsilon^{3/2}}{J^{3/2}}\, n^{-1/2}. \tag{33}$$

Moreover,

$$M \leq \left(\frac{4J^4}{\epsilon^4}\right)^J. \tag{34}$$

Now, let $i, j \in \{1, \ldots, M\}$. We will show that the values of T_1 over the block $V_i \times V_j$ differ by at most 8ϵ. To see this, let $a, c \in V_i$ and $b, d \in V_j$. Then

$$\begin{aligned}
(T_1)_{ab} - (T_1)_{cd} &= \left| \sum_{i<J} \lambda_i \mathbf{u}_i(a)\mathbf{u}_i(b) - \lambda_i \mathbf{u}_i(c)\mathbf{u}_i(d) \right| \\
&\leq \sum_{i<J} |\lambda_i| |\mathbf{u}_i(a)\mathbf{u}_i(b) - \mathbf{u}_i(c)\mathbf{u}_i(b) + \mathbf{u}_i(c)\mathbf{u}_i(b) - \mathbf{u}_i(c)\mathbf{u}_i(d)| \\
&\leq \sum_{i<J} n|\mathbf{u}_i(b)||\mathbf{u}_i(a) - \mathbf{u}_i(c)| + n|\mathbf{u}_i(c)||\mathbf{u}_i(b) - \mathbf{u}_i(d)| \\
&\leq J\left(n \cdot 2\sqrt{2}\sqrt{\frac{J}{\epsilon}}\, n^{-1/2} \cdot \sqrt{2}\frac{\epsilon^{3/2}}{J^{3/2}}\, n^{-1/2} + n \cdot 2\sqrt{2}\sqrt{\frac{J}{\epsilon}}\, n^{-1/2} \cdot \sqrt{2}\frac{\epsilon^{3/2}}{J^{3/2}}\, n^{-1/2}\right) \\
&= 2J\left(n \cdot 2\sqrt{2}\sqrt{\frac{J}{\epsilon}}\, n^{-1/2} \cdot \sqrt{2}\frac{\epsilon^{3/2}}{J^{3/2}}\, n^{-1/2}\right) \\
&< 8\epsilon.
\end{aligned}$$

As a result, we can conclude that, if d_{ij} is the mean of the entries in the block $V_i \times V_j$, then by the triangle inequality and Cauchy-Schwarz,

$$\begin{aligned}
|\mathbf{1}_B^*(T_1 - d_{ij}\mathbf{I})\mathbf{1}_A| &\leq \sum_{a\in A}\sum_{b\in B} |(T_1)_{ab} - d_{ij}| \\
&< 8\epsilon|A||B| \\
&\leq 8\epsilon|V_i||V_j|.
\end{aligned}$$

Next we consider T_2 and observe that $\mathrm{tr}(T_2^2) = \sum_{J\leq j\leq F(J)} \lambda_i^2 \leq \epsilon^3 n^2$.

So, $\sum_{a,b\in V} |(T_2)_{ab}|^2 \leq \epsilon^3 n^2$. Define Σ_1 so that for every $(i,j) \notin \Sigma_1$,

$$\sum_{a\in V_i}\sum_{b\in V_j} |(T_2)_{ab}|^2 \leq \epsilon|V_i||V_j|. \tag{35}$$

Thus,

$$\epsilon^2 \sum_{(i,j)\in\Sigma_1} |V_i||V_j| \leq \sum_{(i,j)\in\Sigma_1}\sum_{a\in V_i}\sum_{b\in V_j} |(T_2)_{ab}|^2 \leq \epsilon^3 n^2.$$

consequently,

$$\sum_{(i,j)\in\Sigma_1} |V_i||V_j| \le \epsilon n^2.$$

So, for any $(i,j) \notin \Sigma_1$, use the fact that $\sum_{a\in V_i}\sum_{b\in V_j} |(T_2)_{ab}|^2 \le \epsilon|V_i||V_j|$ we use Cauchy-Schwarz to obtain the following bound

$$\begin{aligned}
|\mathbf{1}_B^* T_2 \mathbf{1}_A|^2 &= \left|\sum_{a\in A}\sum_{b\in B}(T_2)_{ab}\right|^2 \\
&\le \left(\sum_{a\in A}\sum_{b\in B}|(T_2)_{ab}|^2\right)|A||B| \\
&= \left(\epsilon^2|V_i||V_j|\right)|A||B| \\
|\mathbf{1}_B^* T_2 \mathbf{1}_A| &\le \epsilon|V_i||V_j|.
\end{aligned}$$

Finally, we turn our attention to T_3. Since the maximum eigenvalue of T_3 is $|\lambda_{F(J)}| \le n/\sqrt{F(J)}$, we have, first by Cauchy-Schwarz,

$$|\mathbf{1}_B^* T_3 \mathbf{1}_A| \le |\lambda_{F(J)}||A||B| \le n^2/\sqrt{F(J)}.$$

Let Σ be the pairs $(i,j) \in \{0,1,\ldots,M\}$ such that either $(i,j)\in\Sigma_1$, $i=0$, $j=0$ or $\min(|V_i|,|V_j|) \le \frac{\epsilon n}{M}$. Thus,

$$\begin{aligned}
\sum_{(i,j)\in\Sigma}|V_i||V_j| &\le \sum_{(i,j)\in\Sigma_1}|V_i||V_j| + 2|V_0||V| + 2\sum_{|V_i|<\epsilon n/M}|V_i||V| \\
&\le \epsilon n^2 + 2\epsilon n\cdot n + 2M\frac{\epsilon n}{M}n \le 5\epsilon n^2. \qquad (36)
\end{aligned}$$

If $(i,j)\notin\Sigma$, then for all $A\subseteq V_i$ and $B\subseteq V_j$,

$$\begin{aligned}
|\mathbf{1}_B^*(T-d_{ij})\mathbf{1}_A| &\le |\mathbf{1}_B^*(T_1 - d_{ij}\mathbf{1}_A| + |\mathbf{1}_B^* T_2\mathbf{1}_A| + |\mathbf{1}_B^* T_3 \mathbf{1}_A| \\
&\le 8\epsilon|V_i||V_j| + \epsilon|V_i||V_j| + n^2/\sqrt{F(J)}.
\end{aligned}$$

Finally, we want to establish that $n^2/\sqrt{F(J)} \le \epsilon|V_i||V_j|$ for $(i,j)\notin\Sigma$. This would establish that

$$|\mathbf{1}_B^*(T-d_{ij})\mathbf{1}_A| \le 10\epsilon|V_i||V_j|. \qquad (37)$$

Because $|V_i|,|V_j| \ge \epsilon n/M$, it is sufficient to show that $F(J)\ge M^4/\epsilon^6$ because that would verify that

$$\frac{n^2}{\sqrt{F(J)}} \le \frac{\epsilon^3 n^2}{M^2} \le \epsilon|V_i||V_j|.$$

By (34), $M \le (4J^4/\epsilon^4)^J$. So the function that suffices is

$$F(x) \ge \frac{1}{\epsilon^6}\left(\frac{4x^4}{\epsilon^4}\right)^{4x}.$$

Note that in this proof we use coefficients of 10ϵ and 5ϵ in (37) and (36), respectively. We can, of course, choose $\epsilon/10$ rather than ϵ but we chose these parameters to make the computations somewhat more transparent. □

References

[1] Frieze, Alan; Kannan, Ravi The regularity lemma and approximation schemes for dense problems. 37th Annual Symposium on Foundations of Computer Science (Burlington, VT, 1996), 1220, IEEE Comput. Soc. Press, Los Alamitos, CA, 1996.