Polychromatic Colorings of Complete Graphs with Respect to 1-, 2-factors and Hamiltonian Cycles

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SIAM DM
June 6, 2018
Polychromatic Coloring

Let $G$ and $H$ be graphs and $C$ a set of colors. Let $\varphi : E(G) \rightarrow C$ (not necessarily properly edge-coloring).

$\varphi$ is an $H$-polychromatic coloring of $G$ if every subgraph of $G$ isomorphic to $H$ contains all colors in $C$.

Example $H = K_3$, $G = K_4$, $C = \{\text{red}, \text{blue}\}$. 
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$H$-polychromatic Number

$\phi$ is a $H$-polychromatic coloring of $G$ with respect to $H$ if every subgraph of $G$ isomorphic to $H$ contains all colors in $C$.

Easier to find $\phi$ with fewer colors.
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Easier to find \( \varphi \) with fewer colors.

\( H \)-polychromatic number of \( G \) is the maximum number of colors \( k \) such that there exists a polychromatic coloring of \( G \) with respect to \( H \) using \( k \) colors. Notation \( \text{poly}_H(G) = k \)

Example

\( \text{poly}_{K_3}(K_4) = 3 \)
Motivation for \textbf{H}-polychromatic Number

Let $Q_d$ be a $d$-dimensional hypercube.

\textbf{Problem}

What is the largest $X \subseteq E(Q_n)$ such that $Q_n[X]$ is $Q_d$-free? $\text{ex}(Q_n, Q_d)$?

Example for $Q_2$ in $Q_3$.
**Motivation for $H$-polychromatic Number**

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![Diagram](image_url)
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Example for $Q_2$ in $Q_3$.

Any color class of any $Q_d$-polychromatic coloring of $Q_n$ gives a lower bound on $|X|$.

$$e(Q_n)(1 - 1/poly_{Q_d}(Q_n)) \leq \text{ex}(Q_n, Q_d)$$
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Known Results

Theorem (Alon, Krech, Szabó 2007)

\[
\binom{d+1}{2} \geq \text{poly}_{Q_d}(Q_n) \geq \begin{cases} 
\frac{(d+1)^2}{4} & \text{if } d \text{ is odd} \\
\frac{d(d+2)}{4} & \text{if } d \text{ is even}
\end{cases}
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Theorem (Offner 2008)

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\]
Edge coloring of $H$ is *rainbow* if no two edges of $H$ receive the same color.

Edge coloring of $G$ is *$H$-anti-ramsey* if no copy of $H$ in $G$ is rainbow.

$ar(G, H)$ is the largest number of colors used in an $H$-anti-Ramsey coloring of $G$.

\[
ar(G, H) \leq \text{ex}(G, H)\]

\[
ar(G, H) \geq \left(1 - \frac{2}{\text{poly}_H(G)}\right) e(G)\]
Polychromatic Coloring of Integers

Let $S \subset \mathbb{Z}$ be finite.

Coloring of $\mathbb{Z}$ is $S$-polychromatic if every translation of $S$ contains all colors.

Example: $S = \{0, 1, 4, 5\}$

All about this during the next talk in this session by John Goldwasser.
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Our Results for This Talk

Let $F_k$ be a $k$-factor and $HC$ be a Hamiltonian Cycle.

**Theorem (AGHLMOTY ’18)**
If $n$ is an even positive integer, then $\text{poly}_{F_1}(K_n) = \lfloor \log_2 n \rfloor$.

**Theorem (AGHLMOTY ’18)**
There exists a constant $c$ such that

$$
\lfloor \log_2 2(n + 1) \rfloor \leq \text{poly}_{F_2}(K_n) \leq \text{poly}_{HC}(K_n) \leq \log_2 n + c.
$$

Exact solution for $\text{poly}_{F_2}(K_n)$ and $\text{poly}_{HC}(K_n)$ by G&H.
Constructions For Lower Bounds

\[ \lfloor \log_2 n \rfloor \leq \text{poly}_{F_1}(K_n) \]

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Upper bound for $\text{poly}_F(K_n)$

- show there is an optimal coloring that has ordering of vertices such that for each fixed vertex $v$ “all edges going to the right have the same color”.

- for ever vertex define *inherited color*, counting argument using majority.
Upper bound for $\text{poly}_{F_1}(K_n)$

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Counting first for poly$_{F_1}(K_n)$

for vertex define *inherited color*
Counting first for \( \text{poly}_{F_1}(K_n) \)

for vertex define *inherited color*

Let \( M_c \) be vertices colored color \( c \in \{1, 2, \ldots\} \).

Feature: \( \forall c \) exists \( i_c \in [n] \) such that \( |M_c \cap \{v_1, \ldots, v_i\}| > i/2 \).
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\[
\sum c |M_c| \leq n = \Rightarrow c \leq \lfloor \log_2 n \rfloor
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Counting first for $\text{poly}_{F_1}(K_n)$

For vertex define *inherited color*

Let $M_c$ be vertices colored color $c \in \{1, 2, \ldots \}$.

Feature: $\forall c$ exists $i_c \in [n]$ such that $|M_c \cap \{v_1, \ldots, v_i\}| > i/2$.

Assume that $i_1 < i_2 < \ldots$. By induction $|M_c| \geq 2^c - 1$.

$$\sum_c |M_c| \leq n \implies c \leq \lfloor \log_2 n \rfloor$$
Ordering the vertices

Take largest ordered initial segment,
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Ordering the vertices $v, u, y_1, y_2, y_3 = w_4, y_4 = w_3, \ldots, y_d, w_1, w_2, \ldots, w_d$

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□
Take largest ordered initial segment, $v$ has maximum monochromatic degree \textbf{(red)} in the rest, exists not red $uv$, $y_iw_i$ cannot be blue, all $uw_i$ are blue and $w_i$ is not in the ordered segment, $u$ has higher mono degree than $v$. 

\[ y_1, y_2, \ldots, y_d \]

\[ w_1, w_2, \ldots, w_d \]
Thank you