Independent sets near the lower bound in bounded degree graphs

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**Definition**

An *independent set* in a graph $G$ is an induced subgraph with no edges.

$\alpha(G)$ is the size of a maximum independent set in $G$.  
$\omega(G)$ is the size of a maximum clique in $G$. 
Trivial lower bound

\( \Delta(G) \) is the maximum degree of \( G \).
\( n \) is the number of vertices of \( G \).

If \( \Delta(G) \leq \Delta \), then \( \alpha(G) \geq \frac{n}{\Delta + 1} \).
**Trivial lower bound**

$\Delta(G)$ is the maximum degree of $G$.

$n$ is the number of vertices of $G$.

If $\Delta(G) \leq \Delta$, then $\alpha(G) \geq \frac{n}{\Delta+1}$.
(tight)

What is $\omega(G) \leq \Delta$?
Theorem (Brooks 1941)

If $\Delta(G) \geq 3$ and $\max(\Delta(G), \omega(G)) \leq \Delta$ then $G$ is $\Delta$-colorable.

Implies $\alpha(G) \geq \frac{n}{\Delta}$.

Tight.
Theorem (Albertson, Bollobás, Tucker 1976)

If \( G \) is connected, \( \Delta(G) \leq \Delta \) and \( \omega(G) \leq \Delta - 1 \), then \( \alpha(G) > \frac{n}{\Delta} \) unless \( G \) is one of the following two exceptions:
**Related results**

**Theorem (Albertson, Bollobás, Tucker 1976)**

If $G$ is connected, $\Delta(G) \leq \Delta$ and $\omega(G) \leq \Delta - 1$, then $\alpha(G) > \frac{n}{\Delta}$ unless $G$ is one of the following two exceptions:

![Graph 1](image1.png)

![Graph 2](image2.png)

**Theorem (King, Lu, Peng 2012)**

If $G$ is connected, $\Delta(G) \leq \Delta$ and $\omega(G) \leq \Delta - 1$, then $\alpha(G) > \frac{n}{\Delta - \frac{2}{67}}$ unless $G$ is one of the two exceptions above.
**Small Surplus** \( k \)

If \( G \) with \( \max(\Delta(G), \omega(G)) \leq \Delta \) is

\[
\text{then } \alpha(G) \leq \frac{n-k}{\Delta} + k.
\]
**Small surplus** $k$

If $G$ with $\max(\Delta(G), \omega(G)) \leq \Delta$ is

then $\alpha(G) \leq \frac{n-k}{\Delta} + k$.

**Problem**

If $\alpha(G) \leq \frac{n}{\Delta} + k$ and $\max(\Delta(G), \omega(G)) \leq \Delta$, does $G$ look like

Are there other candidates for $K_\Delta$?
**\(\Delta\)-TIGHT GRAPHS**

A graph is \(\Delta\)-tight if it is \(K_\Delta\) or one of

If \(G\) is \(\Delta\)-tight, then \(\alpha(G) = \frac{n}{\Delta}\).
Our result

Theorem (Dvořák, L.)

Let $\Delta \geq 3$ and $k \geq 0$.
Let $G$ be an $n$-vertex graph with $\max(\Delta(G), \omega(G)) \leq \Delta$.
If $\alpha(G) < \frac{n}{\Delta} + k$, then there exists $X \subseteq V(G)$ of size $< 34\Delta^2 k$ such that $G - X$ is $\Delta$-tightly partitioned.
We will try a sketch for $\Delta = 5$. 
**Δ-FREE VERTICES**

**Definition**
A vertex $v$ of $G$ is $Δ$-free if $v$ is not in $Δ$-tight subgraph.
Many $\Delta$-free vertices $\Rightarrow$ large $\alpha$

**Lemma**

If $\max(\Delta(G), \omega(G)) \leq \Delta$ and $G$ has at least $m$ vertices that are $\Delta$-free then $\alpha(G) \geq \frac{n}{\Delta} + \frac{1}{34\Delta^2} m$. 
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By induction (now only for $\Delta \geq 5$).
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By induction (now only for $\Delta \geq 5$).

- $G$ contains $C_5 \boxtimes P_2$ (hence $\Delta = 5$)

Let $H = G - C_5 \boxtimes P_2$

$\alpha(G) = \alpha(C_5 \boxtimes P_2) + \alpha(H) = 2 + \frac{n-10}{\Delta} + \frac{1}{34\Delta^2} m = \frac{n}{\Delta} + \frac{1}{34\Delta^2} m$
Many $\Delta$-free vertices $\Rightarrow$ large $\alpha$

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By induction (now only for $\Delta \geq 5$).

- $\omega(G) \leq \Delta - 1$ then $\alpha(G) \geq \frac{n}{\Delta} + \frac{n}{34\Delta^2}$ by King, Lu, Peng.
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- $G$ contains $K_\Delta$. 
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![Diagram of a complete graph](image)
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By induction (now only for \( \Delta \geq 5 \)).

- \( G \) contains \( K_\Delta \).

\[
\alpha(G) = \alpha(K_\Delta) + \alpha(G - K_\Delta) \geq 1 + \frac{n - \Delta}{\Delta} + \frac{1}{34\Delta^2} m
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By induction (now only for $\Delta \geq 5$).

- $G$ contains $K_\Delta$.

\[
\alpha(G) = \alpha(K_\Delta) + \alpha(G - K_\Delta) \geq 1 + \frac{n - \Delta}{\Delta} + \frac{1}{34\Delta^2} (m - \Delta)
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By induction (now only for $\Delta \geq 5$).
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No blue triangles.
Many $\Delta$-free vertices $\Rightarrow$ large $\alpha$

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By induction (now only for $\Delta \geq 5$).

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**Lemma**

If \( \max(\Delta(G), \omega(G)) \leq \Delta \) and \( G \) has at least \( m \) vertices that are \( \Delta \)-free then \( \alpha(G) \geq \frac{n}{\Delta} + \frac{1}{34\Delta^2} m \).

By induction (now only for \( \Delta \geq 5 \)).

- \( G \) contains \( K_\Delta \).

Contradiction with \( \Delta(G) \leq \Delta \).
Lemma

An $n$-vertex graph $G$ with $\max(\Delta(G), \omega(G)) \leq \Delta$ can be partitioned into sets $A$, $B$, $C$, and $D$ in time $O(\Delta^2 n)$, such that

- $G[A]$ is $\Delta$-tightly partitioned,
- $G[B]$ is $K_\Delta$-partitioned and $|B| \leq 3\Delta(|C| + |D|)$,
- $C$ is $\Delta$-profitably nibbled,
- $D$ is $\Delta$-free in $G - C$, and
- $\alpha(G) = \alpha(G[B \cup C \cup D]) + |A|/\Delta$. 

\[\text{\includegraphics[width=\textwidth]{partition_lemma_diagram.png}}\]
Counting lemma

Lemma

If $\alpha(G) < \Delta/n + k$, then $|C| + |D| < 34\Delta^2k$. 
Theorem (Dvořák, L.)

If $\alpha(G) < \frac{n}{\Delta} + k$, then exists $X \subseteq V(G)$ of size $< 34\Delta^2k$ such that $G - X$ is $\Delta$-tightly partitioned.

Previous Lemma:

If $\alpha(G) < \frac{n}{\Delta} + k$, then $|C| + |D| < 34\Delta^2k$.

Put $X = C \cup D$. 
Computational Consequences

Computing \( \alpha(G) \) is NP-complete.

If \( \alpha(G) < n/\Delta + k \), can you compute \( \alpha(G) \) efficiently?
Computational Consequences

Computing $\alpha(G)$ is NP-complete.

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YES!
Computational Consequences

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If $\alpha(G) < n/\Delta + k$, can you compute $\alpha(G)$ efficiently?

YES!

• Find partition with $|B \cup C \cup D| < 114\Delta^3k$ in time $O(\Delta^2 n)$.  

\[ \begin{align*}  
A & \quad B & \quad C & \quad D 
\end{align*} \]
Computational Consequences

Computing $\alpha(G)$ is NP-complete.

If $\alpha(G) < n/\Delta + k$, can you compute $\alpha(G)$ efficiently?

YES!

- Find partition with $|B \cup C \cup D| < 114\Delta^3k$ in time $O(\Delta^2 n)$.

- Compute $\alpha(B \cup C \cup D)$ in time $2^O(\Delta^3 k)$. 
Computing $\alpha(G)$ is NP-complete.

If $\alpha(G) < n/\Delta + k$, can you compute $\alpha(G)$ efficiently?

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- Find partition with $|B \cup C \cup D| < 114\Delta^3 k$ in time $O(\Delta^2 n)$.

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- Result is $\alpha(B \cup C \cup D) + \frac{|A|}{\Delta}$.
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- Compute $\alpha(B \cup C \cup D)$ in time $2^{O(\Delta^3 k)}$.

- Result is $\alpha(B \cup C \cup D) + \frac{|A|}{\Delta}$.

- Total time is $2^{O(\Delta^3 k)} + O(\Delta^2 n)$. Efficient if $\Delta$ and $k$ fixed.
Thank you for your attention!