Upper bounds on small Ramsey numbers

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**Definition**

\( R(G_1, G_2, \ldots, G_k) \) is the smallest integer \( n \) such that any \( k \)-edge coloring of \( K_n \) contains a copy of \( G_i \) in color \( i \) for some \( 1 \leq i \leq k \).

\[
R(K_3, K_3) > 5 \\
R(K_3, K_3) \leq 6
\]
**Definition**

$R(G_1, G_2, \ldots, G_k)$ is the smallest integer $n$ such that any $k$-edge coloring of $K_n$ contains a copy of $G_i$ in color $i$ for some $1 \leq i \leq k$.

\[ R(K_3, K_3) > 5 \quad R(K_3, K_3) \leq 6 \]
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R(K_3, K_3) > 5 \\
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\]
Theorem (Ramsey 1930)

$R(K_m, K_n)$ is finite.
**Theorem (Ramsey 1930)**

\[ R(K_m, K_n) \text{ is finite.} \]

\[ R(G_1, \ldots, G_k) \text{ is finite} \]

Questions:
- study how \( R(G_1, \ldots, G_k) \) grows if \( G_1, \ldots, G_k \) grow (large)
- study \( R(G_1, \ldots, G_k) \) for fixed \( G_1, \ldots, G_k \) (small)
**Theorem (Ramsey 1930)**

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Questions:

- study how \( R(G_1, \ldots, G_k) \) grows if \( G_1, \ldots, G_k \) grow (large)
- study \( R(G_1, \ldots, G_k) \) for fixed \( G_1, \ldots, G_k \) (small)
Seminal paper:
David P. Robbins Prize by AMS for Razborov in 2013
Flag algebras

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**Example (Goodman, Razborov)**
If density of edges is at least $\rho > 0$, what is the minimum density of triangles?

- designed to attack extremal problems.
- works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs)
- the results are in limit (very large graphs)
### Applications (Incomplete List)

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<th>Application/Result</th>
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<td>2008</td>
<td>edge density vs. triangle density</td>
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<tr>
<td>Hladký, Král, Norin</td>
<td>2009</td>
<td>Bounds for the Caccetta-Haggvist conjecture</td>
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<td>Razborov</td>
<td>2010</td>
<td>On 3-hypergraphs with forbidden 4-vertex configurations</td>
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<tr>
<td>Hatami, Hladký, Král, Norin, Razborov / Grzesik</td>
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<td>Hatami, Hladký, Král, Norin, Razborov</td>
<td>2012</td>
<td>Non-Three-Colourable Common Graphs Exist</td>
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<td>Balogh, Hu, L., Liu / Baber</td>
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<td>4-cycles in hypercubes</td>
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<td>Reiher</td>
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<td>edge density vs. clique density</td>
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<td>Shagnik, Huang, Ma, Naves, Sudakov</td>
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<td>minimum number of $k$-cliques</td>
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<td>Baber, Talbot</td>
<td>2013</td>
<td>A Solution to the 2/3 Conjecture</td>
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<tr>
<td>Falgas-Ravry, Vaughan</td>
<td>2013</td>
<td>Turán density of many 3-graphs</td>
</tr>
<tr>
<td>Cummings, Král, Pfender, Sperfeld, Treglown, Young</td>
<td>2013</td>
<td>Monochromatic triangles in 3-edge colored graphs</td>
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<tr>
<td>Kramer, Martin, Young</td>
<td>2013</td>
<td>Boolean lattice</td>
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<tr>
<td>Balogh, Hu, L., Pikhurko, Udvari, Volec</td>
<td>2013</td>
<td>Monotone permutations</td>
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<td>Norin, Zwols</td>
<td>2013</td>
<td>New bound on Zarankiewicz’s conjecture</td>
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<tr>
<td>Huang, Linial, Naves, Peled, Sudakov</td>
<td>2014</td>
<td>3-local profiles of graphs</td>
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<td>Balogh, Hu, L., Pfender, Volec, Young</td>
<td>2014</td>
<td>Rainbow triangles in 3-edge colored graphs</td>
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<tr>
<td>Balogh, Hu, L., Pfender</td>
<td>2014</td>
<td>Induced density of $C_5$</td>
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<td>Goaoc, Hubard, de Verclos, Séréni, Volec</td>
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<td>Coregliano, Razborov</td>
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<td>Tournaments</td>
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<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Applications to graphs, oriented graphs, hypergraphs, hypercubes, permutations, crossing number of graphs, order types, geometry, ...

Razborov: Flag Algebra: an Interim Report
**Inspiration**

**Theorem (Cummings, Král, Pfender, Sperfeld, Treglown, Young)**

In every 3-edge-colored complete graph on $n$ vertices, there are at least $\frac{1}{25} \binom{n}{3} + o(n^3)$ monochromatic triangles.

\[
\begin{align*}
\begin{array}{ccc}
\end{align*}
\end{align*}
\]
**Inspiration**

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*In every 3-edge-colored complete graph on $n$ vertices, there are at least $\frac{1}{25} \binom{n}{3} + o(n^3)$ monochromatic triangles.*

\[ \geq \frac{1}{25} \]
Theorem (Cummings, Král, Pfender, Sperfeld, Treglown, Young)

In every 3-edge-colored complete graph on $n$ vertices, there are at least \( \frac{1}{25} \binom{n}{3} + o(n^3) \) monochromatic triangles.

\[
\begin{align*}
&\geq \frac{1}{25} \quad \text{subject to} \quad \begin{align*}
&= 0 \quad \begin{align*}
&= 0
\end{align*}
\end{align*}
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\]
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*In every 3-edge-colored complete graph on \( n \) vertices, there are at least \( \frac{1}{25} \binom{n}{3} + o(n^3) \) monochromatic triangles.*

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\quad + \\
\quad + \\
\quad \geq \frac{1}{25}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{array}{c}
\begin{array}{c}
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\quad \geq \frac{1}{25} \quad \text{subject to} \quad \begin{array}{c}
\quad = \\
\quad = \\
\quad = 0
\end{array}
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\end{array}
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\[
\frac{1}{25} \geq 0
\]
**Inspiration**

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In every 3-edge-colored complete graph on $n$ vertices, there are at least $\frac{1}{25} \binom{n}{3} + o(n^3)$ monochromatic triangles.

\[ \begin{align*}
\geq \frac{1}{25} 
\end{align*} \]

subject to

\[ \begin{align*}
\geq \frac{1}{5} 
\end{align*} \]
Example

What is number of non-edges in a blow-up?
What is number of non-edges in a blow-up?

\[
\sum_{i=1}^{5} \binom{|I_i|}{2} \geq \sum_{i=1}^{5} \binom{n/5}{2} \geq 5 \binom{n/5}{2} \approx \frac{1}{5} \binom{n}{2}
\]
What is number of non-edges in a blow-up?

\[ \sum_{i=1}^{5} \binom{|I_i|}{2} \geq \sum_{i=1}^{5} \binom{n/5}{2} \geq 5 \binom{n/5}{2} \approx \frac{1}{5} \binom{n}{2} \]

**Observation (Key Observation)**

*If a Ramsey graph $G$ has $k$ vertices, then the density of non-edges in any blow-up of $G$ is at least $\frac{1}{k} + o(1)$.***
OUTLINE OF IDEA

OBSERVATION (KEY OBSERVATION)

If a Ramsey graph $G$ has $k$ vertices, then the density of non-edges in any blow-up of $G$ is at least $\frac{1}{k} + o(1)$.
**Outline of Idea**

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If a Ramsey graph $G$ has $k$ vertices, then the density of non-edges in any blow-up of $G$ is at least $\frac{1}{k} + o(1)$.

- Let $\mathcal{G}$ be 2-edge-colored complete graphs with no monochromatic triangle.
- Consider all blow-ups $\mathcal{B}$ of graphs in $\mathcal{G}$
- $\forall B \in \mathcal{B}$, density of non-edges in $B$ is at least $\frac{1}{k} = \frac{1}{5}$.
Outline of idea

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Observation

If density of non-edges $\rho$ is $> \frac{1}{k+1}$ over all $B \in \mathcal{B}$, then Ramsey graph has as most $k$ vertices.
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*If density of non-edges $\rho$ is $> \frac{1}{k+1}$ over all $B \in \mathcal{B}$, then Ramsey graph has as most $k$ vertices.*

If one can prove $\rho > \frac{1}{6}$, then there is no Ramsey graphs on 6 vertices.
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Notice that any lower bound on $\rho$ in $(\frac{1}{k+1}, \frac{1}{k}]$ gives Ramsey graph has at most $k$ vertices.
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- $\forall B \in B$, density of non-edges in $B$ is at least $\frac{1}{k} = \frac{1}{5}$.

$$R(G_1, \ldots, G_n) \leq 1 + \frac{1}{\rho}$$

OBSERVATION

If density of non-edges $\rho$ is $> \frac{1}{k+1}$ over all $B \in B$, then Ramsey graph has as most $k$ vertices.

If one can prove $\rho > \frac{1}{6}$, then there is no Ramsey graphs on 6 vertices.

Notice that any lower bound on $\rho$ in $\left( \frac{1}{k+1}, \frac{1}{k} \right]$ gives Ramsey graph has at most $k$ vertices.
Blow-ups in Flag Algebra

How to characterize blow-ups $B$ of graphs with no $\triangle$, $\triangle$?
**Blow-ups in Flag Algebra**

How to characterize blow-ups $B$ of graphs with no $\triangle$, $\triangle$?

Forbidden subgraphs:

$\triangle$, $\triangle$, $\triangledown$, $\triangledown$
**Blow-ups in Flag Algebra**

How to characterize blow-ups $\mathcal{B}$ of graphs with no $\begin{array}{c} \text{\includegraphics[width=0.05\textwidth]{triangle}} \\ \text{\includegraphics[width=0.05\textwidth]{triangle}} \end{array}$?

Forbidden subgraphs:

![Forbidden subgraphs](image)

minimize

subject to $\begin{array}{c} \text{\includegraphics[width=0.05\textwidth]{triangle}} = \text{\includegraphics[width=0.05\textwidth]{triangle}} = \text{\includegraphics[width=0.05\textwidth]{triangle}} = 0 \end{array}$
How to characterize blow-ups $B$ of graphs with no $\bigtriangleup$, $\bigtriangleup$?

Forbidden subgraphs:

\[ \begin{align*} I_1 & \quad I_2 & \quad I_3 & \quad I_4 & \quad I_5 \\ \begin{array}{c} \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \end{array} & \begin{array}{c} \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \end{array} & \begin{array}{c} \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \end{array} & \begin{array}{c} \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \end{array} & \begin{array}{c} \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \end{array} \end{align*} \]

Flag Algebra question! Easy to modify.

\[ \begin{align*} \text{minimize} & \quad \begin{array}{c} \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \end{array} \\ \text{subject to} & \quad \begin{array}{c} \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \end{array} = \begin{array}{c} \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \end{array} = \begin{array}{c} \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \end{array} = \begin{array}{c} \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \end{array} = \begin{array}{c} \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \end{array} = 0 \end{align*} \]
### New upper bounds (so far)

<table>
<thead>
<tr>
<th>Problem</th>
<th>Lower</th>
<th>New upper</th>
<th>Old upper</th>
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<tr>
<td>$R(K_4^-, K_4^-, K_4^-)$</td>
<td>28</td>
<td>28</td>
<td>30</td>
</tr>
<tr>
<td>$R(K_3, K_4^-, K_4^-)$</td>
<td>21</td>
<td>23</td>
<td>27</td>
</tr>
<tr>
<td>$R(K_4, K_4^-, K_4^-)$</td>
<td>33</td>
<td>47</td>
<td>59</td>
</tr>
<tr>
<td>$R(K_4, K_4, K_4^-)$</td>
<td>55</td>
<td>104</td>
<td>113</td>
</tr>
<tr>
<td>$R(C_3, C_5, C_5)$</td>
<td>17</td>
<td>18</td>
<td>21?</td>
</tr>
<tr>
<td>$R(K_4, K_7^-)$</td>
<td>37</td>
<td>52</td>
<td>59</td>
</tr>
<tr>
<td>$R(K_2,2,2, K_2,2,2)$</td>
<td>30</td>
<td>32</td>
<td>60?</td>
</tr>
<tr>
<td>$R(K_5^-, K_6^-)$</td>
<td>31</td>
<td>38</td>
<td>39</td>
</tr>
<tr>
<td>$R(K_5, K_6^-)$</td>
<td>43</td>
<td>62</td>
<td>67</td>
</tr>
</tbody>
</table>
Example of Computation

Lemma

\[ R(K_3, K_3) \leq 6 \]
**Example of Computation**

**Lemma**

\[ R(K_3, K_3) \leq 6 \]

Our goal is to show:

\[ > \frac{1}{6} \text{ subject to } = = = = = = = 0 \]
**Example of Computation**

**Lemma**

\[ R(K_3, K_3) \leq 6 \]

Our goal is to show:

\[ \frac{1}{6} \text{ subject to } \begin{array}{c}
\begin{array}{c}
\text{ } = \text{ } = \text{ } = \text{ } = \text{ } = 0
\end{array}
\end{array} \]

We show perhaps the most complicated proof of the lemma!
Our goal is to show:

\[ \frac{1}{6} > \text{subject to} \]
Our goal is to show:

\[ > \frac{1}{6} \text{ subject to } \begin{array}{c}
\begin{array}{c}
\text{red triangle} \\
\text{blue triangle} \\
\text{red V} \\
\text{blue V} \\
\text{blue edge} \\
\text{red edge}
\end{array}
\end{array} = 0 \]

Observe that \( \) and \( \) can be swapped.
Our goal is to show:

\[ \frac{1}{6} > \text{ subject to} \]

\[ \begin{align*}
\text{\textbullet} & = \text{\textbullet} = \text{\textbullet} = \text{\textbullet} = \text{\textbullet} = \text{\textbullet} = 0
\end{align*} \]

Observe that \[ \text{\textbullet} \] and \[ \text{\textbullet} \] can be swapped. Change to a colorblind setting. \[ \text{\textbullet} \] is a monochromatic triangle (red or blue).
Our goal is to show:

\[ \frac{1}{6} \text{ subject to } \begin{align*}
\end{align*} \begin{align*}
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Observe that \[ \text{and } \begin{align*}
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Our new goal is to show:

\[ \frac{1}{6} \text{ subject to } \begin{align*}
\end{align*} \begin{align*}
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Our goal is to show:

\[ > \frac{1}{6} \text{ subject to } \begin{align*}
\triangle &= \triangle \\
&= \triangle \\
&= \triangle \\
&= \triangle \\
&= \triangle \\
&= 0
\end{align*} \]

Observe that \( \triangle \) and \( \square \) can be swapped. Change to a colorblind setting. \( \triangle \) is a monochromatic triangle (red or blue).

Our new goal is to show:

\[ > \frac{1}{6} \text{ subject to } \begin{align*}
\triangle &= \triangle \\
&= \triangle \\
&= \triangle \\
&= \triangle \\
&= \triangle \\
&= 0
\end{align*} \]

Colorblind setting will allow to fit the computation on these slides. Also important for bigger applications.
Our goal is to show:

\[
\begin{align*}
    \left\langle \frac{1}{6} \right\rangle & \quad \text{subject to} \quad \begin{array}{ccc}
        & \bullet & \\
        \bullet & & \\
    \end{array} = \begin{array}{ccc}
        & \bullet & \\
        & & \\
        \bullet & & \\
    \end{array} = \begin{array}{ccc}
        & & \\
        & & \\
        & & \\
    \end{array} = 0
\end{align*}
\]
Our goal is to show:

\[ \frac{1}{6} \text{ subject to } \begin{array}{cccc} & & & \end{array} = 0 \]

Basic equations:

\[ \frac{1}{6} \left( \begin{array}{cccc} & & & \end{array} + 0 + 0 + 1 + 3 + 2 + 6 \right) = 1 \]

\[ \frac{1}{6} \left( \begin{array}{cccc} & & & \end{array} \right) = 1 \]
We use flags with type $\sigma_1$ of size two

$$F = \begin{pmatrix} \begin{array}{c} \end{array} & \begin{array}{c} \end{array} & \begin{array}{c} \end{array} \end{pmatrix}^T.$$

For a positive semidefinite matrix $M$

$$0 \leq \begin{bmatrix} F^T MF \end{bmatrix}_{\sigma_1} = \begin{bmatrix} F^T \begin{pmatrix} 0.0744 & -0.0223 & -0.0520 \\ -0.0223 & 0.0238 & -0.0014 \\ -0.0520 & -0.0014 & 0.0536 \end{pmatrix} F \end{bmatrix}_{\sigma_1}$$

$$= -0.0116 \begin{array}{c} \end{array} - 0.3568 \begin{array}{c} \end{array} - 0.1784 \begin{array}{c} \end{array} - 0.0112 \begin{array}{c} \end{array} + 0.3216 \begin{array}{c} \end{array} + 0 \begin{array}{c} \end{array} + 0 \begin{array}{c} \end{array}.$$
\[ \cdot = \frac{1}{6} \left( \begin{array}{c}
1 \\
+ 0 \\
+ 0 \\
+ 1 \\
+ 3 \\
+ 2 \\
+ 6
\end{array} \right) \]

\[ 0 \geq 0.0116 + 0.3568 + 0.1784 + 0.0112 - 0.3216 + 0 + 0 \]
We sum the equations and obtain

\[ 0 \geq 0.1782 + 0.3568 + 0.1784 + 0.1778 + 0.1784 + 0.33 + \ldots > 0.17 > \frac{1}{6}. \]
\[ \frac{1}{6} = \frac{1}{6} \left( 1 + 0 + 0 + 1 + 3 + 2 + 6 \right). \]

\[
0 \geq 0.0116 + 0.3568 + 0.1784 + 0.0112 - 0.3216 + 0 + 0.
\]

We sum the equations and obtain

\[
\geq 0.1782 + 0.3568 + 0.1784 + 0.1778 + 0.1784 + 0.33 + \frac{1}{6}.
\]

Note that the matrix \( M \) was not unique or tight (easy rounding).
(bound \( \geq \frac{1}{5} \) is obtainable)
### What have we tried (so far)

<table>
<thead>
<tr>
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<th>Upper</th>
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<td>27</td>
<td>23</td>
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Thank you for your attention!