Flag Algebras and Applications to Permutations

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Outline

- Extremal problems
- Introduction to the use of Flag Algebras
- Example of automated Flag Algebras approach
- Applications of Flag Algebras in permutations
Problem

What is the minimum number of monotone subsequences of size $k$ in a permutation of $[n]$?
Definition
A graph $G$ consists of vertices $V$ and edges $E \subseteq \binom{V}{2}$.

\[ K_2 \quad K_3 \quad K_4 \quad C_4 \]
Theorem (Mantel 1907)

If a graph $G$ on $n$ vertices has more than $\frac{1}{4} n^2$ edges, then $G$ contains a triangle.
**Theorem (Mantel 1907)**

If a graph $G$ on $n$ vertices has more than $\frac{1}{4}n^2$ edges, then $G$ contains a triangle.

$|E| = 9$
**Theorem (Mantel 1907)**

If a graph $G$ on $n$ vertices has more than $\frac{1}{4}n^2$ edges, then $G$ contains a triangle.

**Theorem (Dirac 1952)**

If all vertices in a graph $G$ on $n$ vertices have degree at least $\frac{n}{2}$, then $G$ is Hamiltonian. ($n \geq 3$)
**Theorem (Mantel 1907)**

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Theorem (Dirac 1952)

If all vertices in a graph $G$ on $n$ vertices have degree at least $\frac{n}{2}$, then $G$ is Hamiltonian. ($n \geq 3$)

Theorem (Erdős, Szekeres 1935)

Every sequence of $n^2 + 1$ distinct numbers contains a monotone subsequence of size $n + 1$. 

$(2,1,4,3,5)$
**Theorem (Mantel 1907)**

If a graph $G$ on $n$ vertices has more than $\frac{1}{4}n^2$ edges, then $G$ contains a triangle.

**Theorem (Dirac 1952)**

If all vertices in a graph $G$ on $n$ vertices have degree at least $\frac{n}{2}$, then $G$ is Hamiltonian. ($n \geq 3$)

**Theorem (Erdős, Szekeres 1935)**

Every sequence of $n^2 + 1$ distinct numbers contains a monotone subsequence of size $n + 1$. 

(2,1,4,3)
Theorem (Mantel 1907)

A triangle-free graph contains at most $\frac{1}{4} n^2$ edges.

Problem

Maximize a graph parameter (\# of edges) over a class of graphs (triangle-free).

- local condition and global parameter
- threshold
- bound and extremal example
Edge-colored graphs

A graph on 5 vertices.
A 2-edge-colored complete graph $K_5$ on 5 vertices.
Densities (ratios) in edge-colored graphs
Densities (ratios) in edge-colored graphs

\[ \frac{7}{10} \]
Densities (ratios) in edge-colored graphs

\[
\begin{align*}
\text{Density} & = \frac{7}{10} \\
\text{Density} & = \frac{3}{10}
\end{align*}
\]
Densities (ratios) in edge-colored graphs

= \frac{7}{10}

= \frac{3}{10}

= \frac{2}{10}
Densities (ratios) in edge-colored graphs

\[
\begin{align*}
&\frac{7}{10} \\
&\frac{3}{10} \\
&\frac{2}{10} \\
&0
\end{align*}
\]
**Theorem (Mantel 1907)**

*A triangle-free graph contains at most* $\frac{1}{4} n^2$ *edges.*

$$= 0$$

$$= \frac{4}{6} = \frac{2}{3} = 0.6$$
**Classical theorems — Mantel’s theorem**

**Theorem (Mantel 1907)**

A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

![Graph diagram]

\[
\begin{align*}
\text{triangle} & \quad = 0 \\
\text{cycle} & \quad = \frac{9}{15} = \frac{3}{5} = 0.6
\end{align*}
\]
Classical theorems — Mantel’s theorem

Theorem (Mantel 1907)

A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

\[
\begin{align*}
16 &= 0 \\
\frac{16}{28} &= \frac{4}{7} = 0.571\ldots
\end{align*}
\]
Classical theorems — Mantel’s theorem

Theorem (Mantel 1907)

A triangle-free graph contains at most $\frac{1}{4} n^2$ edges.
Classical theorems — Mantel’s theorem

Theorem (Mantel 1907)

A triangle-free graph contains at most \( \frac{1}{4} n^2 \) edges.

\[ = 0 \]

\[ = \frac{1}{2} + o(1) \]
**Flag algebras**

Seminal paper:
David P. Robbins Prize by AMS for Razborov in 2013
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David P. Robbins Prize by AMS for Razborov in 2013

Applications to oriented graphs, hypergraphs, crossing number of graphs, geometry, . . .

**Theorem (Hatami, Hladký, Král, Norine, Razborov 2011; Grzesik 2011)**
The number of $C_5$’s in a triangle-free graph on $n$ vertices is at most $(n/5)^5$. 

\[
\begin{array}{c}
\frac{n}{5} \\
\frac{n}{5} \\
\frac{n}{5} \\
\frac{n}{5} \\
\frac{n}{5} \\
\end{array}
\]
Let $G$ be a 2-edge-colored complete graph on $n$ vertices.

The probability that three random vertices in $G$ span a red triangle.
Let $G$ be a 2-edge-colored complete graph on $n$ vertices.

The probability that three random vertices in $G$ span a red triangle.

The probability that three random vertices in $G$ span a triangle with one red and two blue edges.
Flag algebras definitions

Let $G$ be a 2-edge-colored complete graph on $n$ vertices.

- The probability that three random vertices in $G$ span a red triangle.

- The probability that three random vertices in $G$ span a triangle with one red and two blue edges.

- The probability that a random vertex other than $v$ is connected to $v \in V(G)$ by a red edge, i.e., the red degree of $v$ divided by $n - 1$. 
Let $G$ be a 2-edge-colored complete graph on $n$ vertices.

The probability that three random vertices in $G$ span a red triangle.

The probability that three random vertices in $G$ span a triangle with one red and two blue edges.

The probability that a random vertex other than $v$ is connected to $v \in V(G)$ by a red edge, i.e., the red degree of $v$ divided by $n - 1$. 

\[
\begin{align*}
\text{Type} = & \quad 1
\end{align*}
\]
Let $G$ be a 2-edge-colored complete graph on $n$ vertices.

The probability that three random vertices in $G$ span a red triangle.

The probability that three random vertices in $G$ span a triangle with one red and two blue edges.

The probability that a random vertex other than $v$ is connected to $v \in V(G)$ by a red edge, i.e., the red degree of $v$ divided by $n - 1$.

$$v + v = 1$$
Flag algebras definitions

Let $G$ be a 2-edge-colored complete graph on $n$ vertices.

The probability that three random vertices in $G$ span a red triangle.

The probability that three random vertices in $G$ span a triangle with one red and two blue edges.

The probability that a random vertex other than $v$ is connected to $v \in V(G)$ by a red edge, i.e., the red degree of $v$ divided by $n - 1$.

\[ + = 1 \]

Type is a flag induced by labeled vertices.
Flag algebras identities

Let $G$ be a 2-edge-colored complete graph on $n$ vertices. Then

\[
\begin{align*}
\triangle &+ \triangle + \triangle + \triangle = 1.
\end{align*}
\]

Same kind as

\[
\begin{align*}
\bullet &+ \bullet = 1.
\end{align*}
\]
Let $G$ be a 2-edge-colored complete graph on $n$ vertices. Then

\[ \frac{3}{3} + \frac{2}{3} + \frac{1}{3} + \frac{0}{3}. \]

Expanded version where pictures mean graphs:

\[ P \left( \begin{array}{c} \text{in } G \end{array} \right) = P \left( \begin{array}{c} \text{in } \end{array} \right) \cdot P \left( \begin{array}{c} \text{in } G \end{array} \right) + P \left( \begin{array}{c} \text{in } \end{array} \right). \]
Let $G$ be a 2-edge-colored complete graph on $n$ vertices. Then

$$v \times v = v + o(1) = v + v + o(1)$$

$o(1)$ as $|V(G)| \to \infty$ (will be omitted on next slides)
Flag algebras identities

Let $G$ be a 2-edge-colored complete graph on $n$ vertices. Then

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{red} \\
v
\end{array}
\end{array}
\times
\begin{array}{c}
\begin{array}{c}
\text{blue} \\
v
\end{array}
\end{array}
&= \begin{array}{c}
\begin{array}{c}
\text{question mark} \\
v
\end{array}
\end{array}
+ o(1)
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{red} \\
v
\end{array}
\end{array}
\times
\begin{array}{c}
\begin{array}{c}
\text{blue} \\
v
\end{array}
\end{array}
&= \frac{1}{2}
\begin{array}{c}
\begin{array}{c}
\text{red} \\
v
\end{array}
\end{array}
+ o(1)
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{red} \\
v
\end{array}
\end{array}
\times
\begin{array}{c}
\begin{array}{c}
\text{red} \\
v
\end{array}
\end{array}
&= \frac{1}{2}
\begin{array}{c}
\begin{array}{c}
\text{red} \\
v
\end{array}
\end{array}
+ \frac{1}{2}
\begin{array}{c}
\begin{array}{c}
\text{red} \\
v
\end{array}
\end{array}
+ o(1)
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{blue} \\
v
\end{array}
\end{array}
\times
\begin{array}{c}
\begin{array}{c}
\text{blue} \\
v
\end{array}
\end{array}
&= \frac{1}{2}
\begin{array}{c}
\begin{array}{c}
\text{blue} \\
v
\end{array}
\end{array}
+ \frac{1}{2}
\begin{array}{c}
\begin{array}{c}
\text{blue} \\
v
\end{array}
\end{array}
+ o(1)
\end{align*}
\]

\[o(1) \text{ as } |V(G)| \to \infty \text{ (will be omitted on next slides)}\]
Flag algebras identities

Let $G$ be a 2-edge-colored complete graph on $n$ vertices. Then

\[
\begin{align*}
\text{v} \times \text{v} & = \bigtriangleup + o(1) = \bigtriangleup + \bigtriangleup + o(1) \\
\text{v} \times \text{v} & = \frac{1}{2} \bigtriangleup + o(1) = \frac{1}{2} \bigtriangleup + \frac{1}{2} \bigtriangleup + o(1)
\end{align*}
\]

\text{v} \times \text{v}: The probability that choosing two vertices $u_1, u_2$ other than $v$ gives red $vu_1$ and blue $vu_2$.

\text{v}: The probability that choosing two different vertices $u_1, u_2$ other than $v$ gives one of $vu_1$ and $vu_2$ is red and the other is blue.

$o(1)$ as $|V(G)| \to \infty$ (will be omitted on next slides)
Flag algebras identities

Let $G$ be a 2-edge-colored complete graph on $n$ vertices. Then

$$\frac{1}{3} = \frac{1}{|V(G)|} \sum_{v \in V(G)}$$
Let $G$ be a 2-edge-colored complete graph on $n$ vertices. Then

\[
\frac{1}{3} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \binom{n}{3} = \sum_{v \in V(G)} \binom{n-1}{2}
\]
Let $G$ be a 2-edge-colored complete graph on $n$ vertices. Then

\[
\begin{align*}
\frac{1}{3} & = \frac{1}{|V(G)|} \sum_{v \in V(G)} \frac{n}{3} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \binom{n-1}{2}
\end{align*}
\]
Let $G$ be a 2-edge-colored complete graph on $n$ vertices. Then

\[
\frac{1}{3} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \binom{n}{3} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \binom{n-1}{2} \]

\[
\binom{n}{3} = \sum_{v \in V(G)} \binom{n-1}{2}
\]

\[
\binom{n}{3} = \frac{1}{3} \sum_{v \in V(G)} \binom{n-1}{2}
\]
Let $G$ be a 2-edge-colored complete graph on $n$ vertices. Then

$$1 = \begin{array}{c}
\begin{array}{cccc}
\text{red} & \text{red} & \text{red} & \text{red} \\
\text{blue} & \text{blue} & \text{blue} & \text{blue}
\end{array}
+ \begin{array}{cccc}
\text{red} & \text{red} & \text{red} & \text{blue} \\
\text{blue} & \text{blue} & \text{blue} & \text{red}
\end{array}
+ \begin{array}{cccc}
\text{red} & \text{red} & \text{blue} & \text{red} \\
\text{blue} & \text{blue} & \text{red} & \text{blue}
\end{array}
+ \begin{array}{cccc}
\text{red} & \text{red} & \text{blue} & \text{blue} \\
\text{blue} & \text{blue} & \text{red} & \text{blue}
\end{array}
\end{array}$$

$$= \frac{3}{3} + \frac{2}{3} + \frac{1}{3} + \frac{0}{3}$$

$$\begin{array}{c}
\begin{array}{c}
\text{red} \\
\text{blue}
\end{array}
\times
\begin{array}{c}
\text{red} \\
\text{blue}
\end{array}
= \begin{array}{c}
\text{red} \\
\text{blue}
\end{array}
+ \begin{array}{c}
\text{red} \\
\text{blue}
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
\text{red} \\
\text{blue}
\end{array}
\times
\begin{array}{c}
\text{red} \\
\text{blue}
\end{array}
= \frac{1}{2} \begin{array}{c}
\text{red} \\
\text{blue}
\end{array}
+ \frac{1}{2} \begin{array}{c}
\text{red} \\
\text{blue}
\end{array}
\end{array}$$

$$\frac{1}{3} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c}
\text{red} \\
\text{blue}
\end{array}$$

$$; \frac{1}{3} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c}
\text{red} \\
\text{blue}
\end{array}$$
First try for Mantel’s theorem

- How to use the equations to prove something
- Gives bounds as well as helps with extremal examples
Example - Mantel’s theorem, 1st try

Theorem (Mantel 1907)

A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.
Assume edges are red and non-edges are blue.
Example - Mantel’s theorem, 1st try

Theorem (Mantel 1907)

A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

Assume $\Delta = 0$. (We want to conclude $\Delta \leq \frac{1}{2}$.)
**Example - Mantel’s theorem, 1st try**

**Theorem (Mantel 1907)**

A triangle-free graph contains at most $\frac{1}{4} n^2$ edges.

Assume edges are red and non-edges are blue.

Assume $\Delta = 0$. (We want to conclude $\frac{1}{2}$.)

\[
0 \leq \left(1 - 2v\right)^2
\]
Example - Mantel’s theorem, 1st try

**Theorem (Mantel 1907)**

A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

Assume $= 0$. (We want to conclude $\leq \frac{1}{2}$.)

$$0 \leq \left(1 - 2v\right)^2 = \left(1 - 4v + 4v + 4v\right)$$

\[\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
\times
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
\]
Example - Mantel's theorem, 1st try

Theorem (Mantel 1907)

A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

Assume $= 0$. (We want to conclude $\leq \frac{1}{2}$.)

$$0 \leq \frac{1}{n} \sum_v \left(1 - 2\right)^2 = \frac{1}{n} \sum_v \left(1 - 4\right) + 4$$
Example - Mantel’s theorem, 1st try

**Theorem (Mantel 1907)**

A triangle-free graph contains at most \( \frac{1}{4} n^2 \) edges.

Assume edges are red and non-edges are blue.

Assume \( \frac{n^2}{4} = 0 \) (We want to conclude \( \leq \frac{1}{2} \)).

\[
0 \leq \frac{1}{n} \sum_v \left( 1 - 2 \cdot \frac{1}{3} \right)^2 = \frac{1}{n} \sum_v \left( 1 - 4 \cdot \frac{1}{3} + 4 \right)
\]

\[
= 1 - 4 \cdot \frac{1}{3} + 4
\]

\[
\frac{1}{3} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \frac{1}{3}
\]

\[
= \frac{1}{|V(G)|} \sum_{v \in V(G)} \frac{1}{3}
\]
Example - Mantel’s theorem, 1st try

**Theorem (Mantel 1907)**

A triangle-free graph contains at most $\frac{1}{4} n^2$ edges.

Assume edges are red and non-edges are blue.

Assume $\sum_v (1 - 2v^2) = 0$. (We want to conclude $\sum_v \leq \frac{1}{2}$.)

$$0 \leq \frac{1}{n} \sum_v \left( 1 - 2v \right)^2 = \frac{1}{n} \sum_v \left( 1 - 4v + 4v + 4v \right)$$

$$= 1 - 4 + \frac{4}{3} + 4$$

$$= \frac{2}{3} + \frac{1}{3} + 4$$
**Example - Mantel’s theorem, 1st try**

**Theorem (Mantel 1907)**

A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

Assume $\neq 0$. (We want to conclude $\leq \frac{1}{2}$.)

$$0 \leq \frac{1}{n} \sum_v \left(1 - 2 vv\right)^2 = \frac{1}{n} \sum_v \left(1 - 4 vv + 4 \begin{array}{c} v \\ v \end{array} + 4 \begin{array}{c} v \\ v \end{array}\right)$$

$$= 1 - 4 \begin{array}{c} v \\ v \end{array} + \frac{4}{3} \begin{array}{c} v \\ v \end{array}$$

\[\begin{array}{c} v \\ v \end{array} = \frac{2}{3} \begin{array}{c} v \\ v \end{array} + \frac{1}{3} \begin{array}{c} v \\ v \end{array}\]
Example - Mantel’s theorem, 1st try

Theorem (Mantel 1907)

A triangle-free graph contains at most \( \frac{1}{4} n^2 \) edges.

Assume edges are red and non-edges are blue

\[
0 \leq \frac{1}{n} \sum_v \left( 1 - 2 \right) = \frac{1}{n} \sum_v \left( 1 - 4 \right) + 4 \leq \frac{1}{2}.
\]
**Example - Mantel’s theorem, 1st try**

**Theorem (Mantel 1907)**

A *triangle-free graph* contains at most \( \frac{1}{4} n^2 \) edges.

Assume edges are red and non-edges are blue.

Assume \( \sum_v (1 - 2v)^2 = 0 \). (We want to conclude \( 0 \leq \frac{1}{2} \).)

\[
0 \leq \frac{1}{n} \sum_v \left( \begin{array}{c} 1 - 2v \\ v \end{array} \right)^2 = \frac{1}{n} \sum_v \left( 1 - 4v + 4 \right) \\
= 1 - 4 + \frac{4}{3} \\
= 1 - 2 - \frac{2}{3} \\
\leq 1 - 2 \\
2 = \frac{4}{3} + \frac{2}{3}
\]
Example - stability for Mantel

Assume $\Delta = 0$ and $\overline{1} = \frac{1}{2}$. Goal is $G = \bullet \quad \bullet$.
Example - stability for Mantel

Assume $\triangle = 0$ and $\square = \frac{1}{2}$. Goal is $G = \bigcirc$.

$0 \leq 1 - 2 \square - \frac{2}{3}$
Example - stability for Mantel

Assume \( \triangle = 0 \) and \( \square = \frac{1}{2} \). Goal is \( G = \).

\[
0 \leq 1 - 2 - \frac{2}{3} \\
0 \leq -\frac{2}{3}
\]
Example - stability for Mantel

Assume $= 0$ and $= \frac{1}{2}$. Goal is $G = \cdot$

$0 \leq 1 - 2 - \frac{2}{3}$

$0 \leq -\frac{2}{3}$

Only and appear.
Example - stability for Mantel

Assume \[\Delta = 0\] and \[\lambda = \frac{1}{2}\]. Goal is \(G = \left\{ \_ \right\}\).

\[
0 \leq 1 - 2 \quad \lambda - \frac{2}{3}
\]

\[
0 \leq -\frac{2}{3}
\]

Only \(\Delta\) and \(\lambda\) appear.
**Example - stability for Mantel**

Assume $\Delta = 0$ and $\mathcal{I} = \frac{1}{2}$. Goal is $G = \begin{array}{c}
\end{array}$.

\[
0 \leq 1 - 2 - \frac{2}{3}
\]

\[
0 \leq -\frac{2}{3}
\]

Only $\Delta$ and $\mathcal{I}$ appear.
Assume $= 0$ and $= \frac{1}{2}$. Goal is $G = \emptyset$. 

$0 \leq 1 - 2 \quad \frac{2}{3}$

$0 \leq -\frac{2}{3}$

Only and appear.
Example - stability for Mantel

Assume \( e = 0 \) and \( \alpha = \frac{1}{2} \). Goal is \( G = \)\( \emptyset \).

\[
0 \leq 1 - 2 - \frac{2}{3}
\]

Only \( e \) and \( \alpha \) appear.
Example - stability for Mantel

Assume $= 0$ and $= \frac{1}{2}$. Goal is $G = \cdot$

\[
0 \leq 1 - 2 - \frac{2}{3} \\
0 \leq -\frac{2}{3}
\]

Only $\bigtriangleup$ and $\bigtriangleup$ appear.
Example - Stability for Mantel

Assume $\Delta = 0$ and $\square = \frac{1}{2}$. Goal is $G = \bigcirc$.

\[
0 \leq 1 - 2 \quad - \quad \frac{2}{3}
\]

\[
0 \leq -\frac{2}{3}
\]

Only $\triangle$ and $\bigtriangleup$ appear.
Flag Algebras - formal approach

- consider 2-edge-colored graphs $G_1, G_2, \ldots$ ($|G_n| \to \infty$)
Flag Algebras - Formal Approach

- consider 2-edge-colored graphs $G_1, G_2, \ldots (|G_n| \to \infty)$
- $p_n(F) :=$ probability that random $|F|$ vertices of $G_n$ induces $F$
Flag Algebras - formal approach

- consider 2-edge-colored graphs $G_1, G_2, \ldots$ ($|G_n| \to \infty$)
- $p_n(F)$ := probability that random $|F|$ vertices of $G_n$ induces $F$
- sequence $(G_n)$ is convergent if $p_n(F)$ converge for all $F$
Flag Algebras - formal approach

- consider 2-edge-colored graphs $G_1, G_2, \ldots$ ($|G_n| \to \infty$)
- $p_n(F) :=$ probability that random $|F|$ vertices of $G_n$ induces $F$
- sequence $(G_n)$ is convergent if $p_n(F)$ converge for all $F$
- limit object – function $q$: all finite 2-edge-colored graphs $\to [0, 1]$
Flag Algebras - formal approach

- consider 2-edge-colored graphs $G_1, G_2, \ldots$ ($|G_n| \to \infty$)
- $p_n(F) :=$ probability that random $|F|$ vertices of $G_n$ induces $F$
- sequence $(G_n)$ is convergent if $p_n(F)$ converge for all $F$
- limit object – function $q$: all finite 2-edge-colored graphs $\to [0, 1]$
- $q$ yields homomorphism from linear combinations of graphs to $\mathbb{R}$
Flag Algebras - Formal Approach

- consider 2-edge-colored graphs $G_1, G_2, \ldots \ (|G_n| \to \infty)$
- $p_n(F) :=$ probability that random $|F|$ vertices of $G_n$ induces $F$
- sequence $(G_n)$ is convergent if $p_n(F)$ converge for all $F$
- limit object – function $q$: all finite 2-edge-colored graphs $\to [0, 1]$
- $q$ yields homomorphism from linear combinations of graphs to $\mathbb{R}$
- the set of limit objects $\text{LIM} = \text{homomorphisms } q: q(F) \geq 0$
Flag Algebras - formal approach

- consider 2-edge-colored graphs $G_1, G_2, \ldots$ ($|G_n| \to \infty$)
- $p_n(F) :=$ probability that random $|F|$ vertices of $G_n$ induces $F$
- sequence $(G_n)$ is convergent if $p_n(F)$ converge for all $F$
- limit object – function $q$: all finite 2-edge-colored graphs $\to [0,1]$
- $q$ yields homomorphism from linear combinations of graphs to $\mathbb{R}$
- the set of limit objects $\text{LIM} = \text{homomorphisms } q: q(F) \geq 0$
- we optimize on $\text{LIM}^T = \left\{ q \in \text{LIM} : q \left( \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \right) = 0 \right\}$

\[ \frac{1}{2} \geq \max_{q \in \text{LIM}^T} q \left( \begin{array}{c} \bullet \\ \bullet \end{array} \right) \]
More automatic approach

- How to use computer to guess the right equation for you.

\[ 0 \leq \left( 1 - 2v \right)^2 \]
Example - Mantel’s theorem, 2nd try

Theorem (Mantel 1907)

A triangle-free graph contains at most \( \frac{1}{4} n^2 \) edges.

Assume edges are red and non-edges are blue.
**Example - Mantel's theorem, 2nd try**

**Theorem (Mantel 1907)**

A triangle-free graph contains at most $\frac{1}{4} n^2$ edges.

Assume edges are red and non-edges are blue.

Assume $\triangle = 0$. (We want to conclude $\leq \frac{1}{2}$.)
**Example - Mantel’s Theorem, 2nd Try**

**Theorem (Mantel 1907)**

A *triangle-free graph contains at most* $\frac{1}{4}n^2$ *edges.*

Assume edges are red and non-edges are blue.

Assume $\begin{array}{c}
\begin{tikzpicture}
  \draw[blue, very thick] (0,0) -- (1,0);
  \draw[red, very thick] (0,0) -- (0,1);
  \draw[red, very thick] (1,0) -- (0,1);
\end{tikzpicture}
\end{array}$ = 0. (We want to conclude $\begin{array}{c}
\begin{tikzpicture}
  \draw[blue, very thick] (0,0) -- (1,0);
  \draw[red, very thick] (0,0) -- (0,1);
  \draw[red, very thick] (1,0) -- (0,1);
\end{tikzpicture}
\end{array}$ $\leq \frac{1}{2}.$)
**Theorem (Mantel 1907)**

A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

Assume $\Delta = 0$. (We want to conclude $\Delta \leq \frac{1}{2}$.)

\[
\Delta = 0 \cdot \triangle + \frac{1}{3} \cdot \triangle + \frac{2}{3} \cdot \triangle \\
\leq \frac{2}{3} \left( \triangle + \triangle + \triangle \right)
\]
**Example - Mantel’s theorem, 2nd try**

**Theorem (Mantel 1907)**

A triangle-free graph contains at most \( \frac{1}{4} n^2 \) edges.

Assume edges are red and non-edges are blue.

Assume \( \triangle = 0 \). (We want to conclude \( \triangle \leq \frac{1}{2} \).)

\[
1 = \triangle + \frac{1}{3} \triangle + \frac{2}{3} \triangle \\
\leq \frac{2}{3} \left( \triangle + \triangle + \triangle \right) 
\]
Example - Mantel’s theorem, 2nd try

Theorem (Mantel 1907)

A triangle-free graph contains at most $\frac{1}{4} n^2$ edges.

Assume edges are red and non-edges are blue.

Assume $\Delta = 0$. (We want to conclude $\Delta \leq \frac{1}{2}$.)

\[
\begin{align*}
1 &= \frac{2}{3} \left( \begin{array}{c}
\Delta_1 \quad + \quad \Delta_2 \quad + \quad \Delta_3 \\
\end{array} \right) \\
\Delta &= 0 + \frac{1}{3} \Delta_1 + \frac{2}{3} \Delta_2 \\
\Delta &\leq \frac{2}{3} \left( \begin{array}{c}
\Delta_1 \quad + \quad \Delta_2 \quad + \quad \Delta_3 \\
\end{array} \right)
\end{align*}
\]
**Example - Mantel’s theorem, 2nd try**

**Theorem (Mantel 1907)**

A *triangle-free graph contains at most* \( \frac{1}{4} n^2 \) *edges.*

Assume edges are red and non-edges are blue.

Assume \( \triangle = 0 \). (We want to conclude \( \triangle \leq \frac{1}{2} \).)

\[
\begin{aligned}
1 &= \triangle + \frac{1}{3} \triangle + \frac{2}{3} \\
\leq \frac{2}{3} \left( \triangle + \triangle + \triangle \right) \\
\leq \frac{2}{3}
\end{aligned}
\]
Assume \( \triangle = 0 \). (We want to conclude \( \frac{1}{2} \).

\[
= 0 \quad + \quad \frac{1}{3} \quad + \quad \frac{2}{3}
\]
Assume $\triangle = 0$. (We want to conclude $\triangle \leq \frac{1}{2}$.)

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph $G$

$0 \leq c_1 \triangle + c_2 \triangle + c_3 \triangle$. 
**Example - Mantel’s theorem, 2nd try**

Assume $0 = 0$. (We want to conclude $\leq \frac{1}{2}$.)

$I = 0 \leq \frac{1}{3} + \frac{2}{3}$

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph $G$

$0 \leq c_1 + c_2 + c_3$.

After summing together

$\leq c_1 + (\frac{1}{3} + c_2) + (\frac{2}{3} + c_3)$

and

$\leq \max \left\{ 0 + c_1, \frac{1}{3} + c_2, \frac{2}{3} + c_3 \right\}$. 

Example - Mantel’s Theorem, 2nd Try

Assume \( \begin{array}{c}
\begin{array}{c}
\text{Assume } \\
\text{= 0. (We want to conclude } \\
\text{\leq } \frac{1}{2}. \\
\end{array}
\end{array} \)

\[ \begin{array}{c}
\begin{array}{c}
\quad = 0 \\
\quad + 1 \\
\quad + 2 \\
\end{array}
\end{array} \]

Idea: find \( c_1, c_2, c_3 \in \mathbb{R} \) such that for every graph \( G \)

\[ 0 \leq c_1 \quad + c_2 \quad + c_3. \]

After summing together

\[ \begin{array}{c}
\begin{array}{c}
\quad \leq c_1 \\
\quad + (\frac{1}{3} + c_2) \\
\quad + (\frac{2}{3} + c_3)
\end{array}
\end{array} \]

and

\[ \begin{array}{c}
\begin{array}{c}
\quad \leq \max \left\{ (0 + c_1), \frac{1}{3} + c_2, \frac{2}{3} + c_3 \right\}
\end{array}
\end{array} \]

\( c_3 < 0 \)
CANDIDATES FOR $c_1$, $c_2$, $c_3$

\[
\begin{pmatrix}
a & c \\
c & b \\
\end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)}
\]
CANDIDATES FOR $c_1$, $c_2$, $c_3$

$$0 \leq \begin{pmatrix} v' & v \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} v' \\ v \end{pmatrix}^T$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)}$$
CANDIDATES FOR $c_1, c_2, c_3$

\[
0 \leq \begin{pmatrix} v \cdot v \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} v \cdot v \end{pmatrix}^T
= a \begin{array}{c} \times \end{array} v + b \begin{array}{c} \times \end{array} v + \frac{1}{2} c \begin{array}{c} \times \end{array} v + \frac{1}{2} c \begin{array}{c} \times \end{array} v
\]

\[
\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)}
\]
CANDIDATES FOR $c_1, c_2, c_3$

$$0 \leq \begin{pmatrix} v', v \\ v, v \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} v', v \\ v, v \end{pmatrix}^T$$

$$= a \begin{array}{c} v \end{array} + b \begin{array}{c} v \end{array} + c \begin{array}{c} v \end{array}$$

$$\begin{array}{c} v \end{array} \times \begin{array}{c} v \end{array} = ?$$

$$\begin{array}{c} v \end{array} \times \begin{array}{c} v \end{array} = \frac{1}{2} \begin{array}{c} v \end{array}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)}$$
**Candidates for** $c_1, c_2, c_3$

\[
0 \leq \begin{pmatrix} v' & v \\ v & v \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} v' & v \\ v & v \end{pmatrix}^T
\]

\[
= a \quad v + b \quad v + c \quad v
\]

\[
\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)}
\]
CANDIDATES FOR $c_1, c_2, c_3$

$$0 \leq \frac{1}{n} \sum_v \left( \begin{array}{cc} v \end{array} \right) \left( \begin{array}{cc} a & c \\ c & b \end{array} \right) \left( \begin{array}{cc} v \end{array} \right)^T$$

$$= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \\ v \end{array} \begin{array}{c} \bullet \end{array} + b \begin{array}{c} \bullet \\ v \end{array} \begin{array}{c} \bullet \end{array} + c \begin{array}{c} \bullet \\ v \end{array}$$

$$\left( \begin{array}{cc} a & c \\ c & b \end{array} \right) \succeq 0 \text{ (matrix is positive semidefinite)}$$
**Candidates for** $c_1, c_2, c_3$

\[
0 \leq \frac{1}{n} \sum_v \left( \begin{pmatrix} v & v \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} v & v \end{pmatrix} \right)^T
\]

\[
= \frac{1}{n} \sum_v \begin{pmatrix} v \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} v \end{pmatrix}
\]

\[
= a \begin{pmatrix} v \end{pmatrix} + \frac{a + 2c}{3} \begin{pmatrix} v \end{pmatrix} + \frac{b + 2c}{3} \begin{pmatrix} v \end{pmatrix} + b \begin{pmatrix} v \end{pmatrix}
\]

\[
\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \left( \begin{pmatrix} v \end{pmatrix} \begin{pmatrix} v \end{pmatrix} \right)
\]

\[
= \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{pmatrix} v \end{pmatrix}
\]

\[
= \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{pmatrix} v \end{pmatrix}
\]

\[
= \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{pmatrix} v \end{pmatrix}
\]
CANDIDATES FOR $c_1, c_2, c_3$

\[
0 \leq \frac{1}{n} \sum_v \begin{pmatrix} \blacklozenge_v & \blacklozenge_v \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} \blacklozenge_v & \blacklozenge_v \end{pmatrix}^T
\]

\[
= \frac{1}{n} \sum_v a \begin{array}{c} \blacklozenge_v \\ \blacklozenge_v \end{array} + b \begin{array}{c} \blacklozenge_v \\ \blacklozenge_v \end{array} + c \begin{array}{c} \blacklozenge_v \\ \blacklozenge_v \end{array}
\]

\[
= a \begin{array}{c} \blacklozenge_v \\ \blacklozenge_v \end{array} + \frac{a + 2c}{3} \begin{array}{c} \blacklozenge_v \\ \blacklozenge_v \end{array} + \frac{b + 2c}{3} \begin{array}{c} \blacklozenge_v \\ \blacklozenge_v \end{array}
\]

\[
\frac{1}{3} \begin{array}{c} \blacklozenge_v \\ \blacklozenge_v \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \blacklozenge_v \\ \blacklozenge_v \end{array}
\]

\[
\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succ 0 \implies \frac{2}{3} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \blacklozenge_v \\ \blacklozenge_v \end{array}
\]
CANDIDATES FOR $c_1, c_2, c_3$

\[ 0 \leq \frac{1}{n} \sum_v \begin{pmatrix} 1 \end{pmatrix}_v \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix}_v^T \]

\[ = \frac{1}{n} \sum_v a + b + c \]

\[ = a + \frac{a + 2c}{3} + \frac{b + 2c}{3} \]

\[ c_1 = a, \quad c_2 = \frac{a + 2c}{3}, \quad c_3 = \frac{b + 2c}{3} \]

\[ \begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)} \]
Using $c_1, c_2, c_3$

\[
\begin{align*}
1 &= \triangle + \frac{1}{3} \triangle + \frac{2}{3} \triangle \\
0 \leq a \triangle + \frac{a + 2c}{3} \triangle + \frac{b + 2c}{3} \triangle
\end{align*}
\]

\[
\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)}
\]
Using $c_1, c_2, c_3$

\[
\begin{align*}
\left( \begin{array}{cc}
   a & c \\
   c & b \\
\end{array} \right) & \succeq 0 \quad \text{(matrix is positive semidefinite)}
\end{align*}
\]
Using $c_1, c_2, c_3$

\[
\begin{align*}
0 \leq a + \frac{a + 2c}{3} + \frac{b + 2c}{3} & \leq \max \left\{ a, \frac{1 + a + 2c}{3}, \frac{2 + b + 2c}{3} \right\}.
\end{align*}
\]

\[
\begin{pmatrix}
a & c \\
c & b
\end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)}
\]
Using $c_1$, $c_2$, $c_3$

\[
\begin{align*}
\text{Try} & \quad \left( \begin{array}{cc} a & c \\ c & b \end{array} \right) = \left( \begin{array}{cc} 1/2 & -1/2 \\ -1/2 & 1/2 \end{array} \right). \\
0 & \leq a + \frac{a + 2c}{3} + \frac{b + 2c}{3} \\
\leq \max \left\{ a, \frac{1 + a + 2c}{3}, \frac{2 + b + 2c}{3} \right\}.
\end{align*}
\]
Using $c_1, c_2, c_3$

\[
\begin{align*}
0 & \leq a + \frac{a+2c}{3} + \frac{b+2c}{3}
\end{align*}
\]

Try

\[
\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.
\]

It gives

\[
\begin{align*}
\leq & \quad \max \left\{ \frac{1}{2}, \frac{1}{6}, \frac{1}{2} \right\} = \frac{1}{2}.
\end{align*}
\]
Optimizing $a, b, c$

\[
\leq \max \left\{ a, \frac{1 + a + 2c}{3}, \frac{2 + b + 2c}{3} \right\}
\]

\[
\begin{aligned}
\text{(SDP)} & \quad \begin{cases}
\text{Minimize} & d \\
\text{subject to} & a \leq d \\
& \frac{1 + a + 2c}{3} \leq d \\
& \frac{2 + b + 2c}{3} \leq d \\
& \left( \begin{array}{cc}
a & c \\
c & b \\
\end{array} \right) \succeq 0
\end{cases}
\end{aligned}
\]

(SDP) can be solved on computers using CSDP or SDPA. Rounding may be needed for exact results.
PERMUTATIONS AND EXTREMAL PROBLEMS

PROBLEM
What is the minimum number of monotone subsequences of size $k$ in a permutation of $[n]$?
**Problem**

*What is the minimum number of monotone subsequences of size $k$ in a permutation of $[n]$?*

$k = 3$

$n = 5$

$(5,4,1),(5,4,2),(5,4,3)$

$(1,2,3)$

$(5,4,1,2,3)$
Permutations and extremal problems

Problem

What is the minimum number of monotone subsequences of size $k$ in a permutation of $[n]$?

$k = 3$
$n = 5$

(5,4,1), (5,4,2), (5,4,3)
(1,2,3)

(5,4,1,2,3)

(1,2,3)

(4,5,1,2,3)
**Conjecture**

**Conjecture (Myers 2002)**

The number of monotone subsequences of length $k$ is minimized by a permutation on $[n]$ with $k - 1$ increasing runs of as equal lengths as possible.

$k = 4, n = 15$
Extremal case is not unique
Extremal problems
Flag algebras
First try for Mantel
More automatic approach
Permutations

Extremal case is not unique
Extremal case is not unique
EXTREMAL CASE IS NOT UNIQUE
Extremal case is not unique
Extremal case is not unique
Extremal case is not unique
Extremal case is not unique
Conjecture (Myers 2002)

The number of monotone subsequences of length \( k \) is minimized by a permutation on \([n]\) with \( k - 1 \) increasing runs of as equal lengths as possible.

Theorem (Samotij, Sudakov ’14+)

Myers’ conjecture is true for sufficiently large \( k \) and 
\[ n \leq k^2 + ck^{3/2} \log k, \] where \( c \) is an absolute positive constant.

Theorem (Balogh, Hu, L., Pikhurko, Udvari, Volec ’14+)

Myers’ conjecture is true for \( k = 4 \) and \( n \) sufficiently large.

\[
\begin{align*}
(1,2,3,4) &\quad \quad (4,3,2,1)
\end{align*}
\]
**Conjecture (Myers 2002)**

The number of monotone subsequences of length \( k \) is minimized by a permutation on \([n]\) with \( k - 1 \) increasing runs of as equal lengths as possible.

**Theorem (Samotij, Sudakov ’14+)**

Myers’ conjecture is true for sufficiently large \( k \) and \( n \leq k^2 + ck^{3/2} \log k \), where \( c \) is an absolute positive constant.

**Theorem (Balogh, Hu, L., Pikhurko, Udvari, Volec ’14+)**

Myers’ conjecture is true for \( k = 4 \) and \( n \) sufficiently large.

\[
\begin{array}{c}
(1,2,3,4) \\
(4,3,2,1)
\end{array}
\]

Use of flag algebras, \( k = 5, 6 \) also doable, 7 not.
From permutations to permutation graphs

(1,2)  (2,1)
FROM PERMUTATIONS TO PERMUTATION GRAPHS

\( k = 3 \)
\( n = 5 \)

\( (1,2) \)
\( (5,4,1,2,3) \)

\( (2,1) \)

\( 1 \)
\( 2 \)
\( 3 \)
\( 4 \)
\( 5 \)
Extremal problems
Flag algebras
First try for Mantel
More automatic approach
Permutations

Extremal example \((k = 4)\)
As flag algebra question \((k = 4)\)

\[
\begin{array}{cc}
\text{(1,2,3,4)} & \text{(4,3,2,1)} \\
\end{array}
\]
As flag algebra question \((k = 4)\)

\[
\begin{align*}
(1,2,3,4) & \quad (4,3,2,1)
\end{align*}
\]
As flag algebra question \((k = 4)\)
As flag algebra question ($k = 4$)

(1,2,3,4) + (4,3,2,1)

minimize $+\quad \geq \quad \frac{1}{27}$

Theorem (Balogh, Hu, L., Pikhurko, Udvari, Volec ’14+)

for every permutation graph.
**Only for permutation graphs**

**Theorem (Balogh, Hu, L., Pikhurko, Udvari, Volec ’14+)**

\[
\min \left( \begin{array}{cc}
1 & 1 \\
1 & 1
\end{array} \right) + \left( \begin{array}{cc}
1 & 1 \\
1 & 1
\end{array} \right) = \frac{1}{27}
\]

over permutation graphs (and extremal permutations described using Myers’ results).
Only for permutation graphs

Theorem (Balogh, Hu, L., Pikhurko, Udvari, Volec ’14+)

$$\min \left(\begin{array}{c}
\begin{array}{c}
1
\end{array}
+ \begin{array}{c}
1
\end{array}
\end{array}\right) = \frac{1}{27}$$

over permutation graphs (and extremal permutations described using Myers’ results).

Theorem (Sperfeld ’12; Thomason ’89)

$$\frac{1}{35} < \min \left(\begin{array}{c}
\begin{array}{c}
1
\end{array}
+ \begin{array}{c}
1
\end{array}
\end{array}\right) < \frac{1}{33}$$

over all sufficiently large 2-edge-colored complete graphs.
We want to prove \( + \geq \frac{1}{27} \).
We want to prove \[ \begin{array}{cc}
\times & + \\
\times & \geq \frac{1}{27}
\end{array} \]

- Write a semidefinite program \((SDP)\) (with graphs on 7 vertices, 388 constraints).
We want to prove \( + \geq \frac{1}{27} \).

- Write a semidefinite program (SDP) (with graphs on 7 vertices, 388 constraints).
- Solve (SDP) using a computer, obtain \( M' \in \mathbb{R}^{f \times f} \).
We want to prove \( \geq \frac{1}{27} = 0.037 \)

- Write a semidefinite program \((SDP)\) (with graphs on 7 vertices, 388 constraints).
- Solve \((SDP)\) using a computer, obtain \( M' \in \mathbb{R}^{f \times f} \).
- \( M' \) gives \( \geq 0.0370370369999 \)
We want to prove \[ \begin{array}{cccc} & & & \bullet \end{array} + \begin{array}{cccc} & & & \bullet \end{array} \geq \frac{1}{27} = 0.037 \]

- Write a semidefinite program (SDP) (with graphs on 7 vertices, 388 constraints).
- Solve (SDP) using a computer, obtain \( M' \in \mathbb{R}^{f \times f} \).
- \( M' \) gives
  \[ \begin{array}{cccc} & & & \bullet \end{array} + \begin{array}{cccc} & & & \bullet \end{array} \geq 0.0370370369999 \]

- Round \( M' \) to \( M \in \mathbb{Q}^{f \times f} \), such that
  \[ \begin{array}{cccc} & & & \bullet \end{array} + \begin{array}{cccc} & & & \bullet \end{array} \geq \frac{1}{27} \]

and \( M \succeq 0 \).
Structure of extremal permutations

Assuming

Flag algebra implies:

\[
\begin{array}{cccc}
\text{Red flag} & \text{Blue flag} & = & \frac{1}{27}
\end{array}
\]
**STRUCTURE OF EXTREMAL PERMUTATIONS**

Assuming

\[
\begin{align*}
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} = \frac{1}{27}
\end{align*}
\]

Flag algebra implies:

\[
(A) \quad \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = o(1)
\]
Structure of extremal permutations

Assuming

\[ + = \frac{1}{27} \]

Flag algebra implies:

(A) \[ = o(1) \]

(B) Almost all are .
After flag algebra (stability)

"+ is close to $\frac{1}{27}$ $\Rightarrow$ $G$ is close to or "

Lemma (Stability)

For every $\varepsilon > 0$ there exist $n_0$ and $\varepsilon' > 0$ such that every admissible graph $G$ of order $n > n_0$ with

\[
\begin{align*}
+ & \leq \frac{1}{27} + \varepsilon' 
\end{align*}
\]

is isomorphic to either

after recoloring at most $20\varepsilon n^2$ edges.
After flag algebra (stability sketch)

“\(\begin{array}{c|c}
\text{red} & \text{blue} \\
\hline
\text{x} & \text{y} \\
\end{array}\) is close to \(\frac{1}{27}\) ⇒ \(G\) is close to \(\begin{array}{c}
\text{red} \\
\hline
\text{blue} \\
\end{array}\) or \(\begin{array}{c}
\text{blue} \\
\hline
\text{red} \\
\end{array}\)”

- Using removal lemma, properties (A) and (B) can be satisfied entirely. (lost \(\varepsilon n^2\) edges)
After flag algebra (stability sketch)

“\(\begin{array}{c}
\text{\textcolor{red}{\textbullet}} & \text{\textcolor{blue}{\textbullet}} \\
\end{array}\) is close to \(\frac{1}{27}\) ⇒ \(G\) is close to \(\begin{array}{c}
\text{\textcolor{red}{\textbullet}} & \text{\textcolor{red}{\textbullet}} \\
\text{\textcolor{red}{\textbullet}} & \text{\textcolor{red}{\textbullet}} \\
\text{\textcolor{red}{\textbullet}} & \text{\textcolor{red}{\textbullet}} \\
\text{\textcolor{red}{\textbullet}} & \text{\textcolor{red}{\textbullet}} \\
\end{array}\) or \(\begin{array}{c}
\text{\textcolor{blue}{\textbullet}} & \text{\textcolor{blue}{\textbullet}} \\
\text{\textcolor{blue}{\textbullet}} & \text{\textcolor{blue}{\textbullet}} \\
\text{\textcolor{blue}{\textbullet}} & \text{\textcolor{blue}{\textbullet}} \\
\text{\textcolor{blue}{\textbullet}} & \text{\textcolor{blue}{\textbullet}} \\
\end{array}\)”

• Using removal lemma, properties (A) and (B) can be satisfied entirely. (lost \(\varepsilon n^2\) edges)

• For all \(v \in V(G) \setminus X\), where \(|X| \leq 2\varepsilon n\) vertices

\[
\frac{1}{27} - \varepsilon \leq \begin{array}{c}
\text{\textcolor{red}{\textbullet}} & \text{\textcolor{red}{\textbullet}} \\
\end{array} \quad v + \begin{array}{c}
\text{\textcolor{blue}{\textbullet}} & \text{\textcolor{blue}{\textbullet}} \\
\end{array} \quad v \leq \frac{1}{27} + \varepsilon'' \quad (1)
\]
After Flag Algebra (Stability Sketch)

"\( + \) is close to \( \frac{1}{27} \Rightarrow G \) is close to or"

- Using removal lemma, properties (A) and (B) can be satisfied entirely. (lost \( \varepsilon n^2 \) edges)
- For all \( v \in V(G) \setminus X \), where \(|X| \leq 2\varepsilon n\) vertices

\[
\frac{1}{27} - \varepsilon \leq \binom{v}{u} \binom{v}{u} \leq \frac{1}{27} + \varepsilon''
\]  \hspace{1cm} (1)

- \( x \sim y \) if

\[ x \quad \leftrightarrow \quad y \]
“$+\quad\Rightarrow G$ is close to $\bigcirc$ or $\bigtriangleup$”

- $x \sim y$ if $x \neq y$

(A) $= 0$

(B) All are.
"\( + \) is close to \( \frac{1}{27} \) \( \Rightarrow \) \( G \) is close to \( \) or \( \)

- \( x \sim y \) if \( x \)

\( (A) \)

\( = 0 \)

\( (B) \)

All are .

- Every equivalence class is a monochromatic clique.
"$\begin{array}{cc}
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\end{array} \text{ is close to } \frac{1}{27} \Rightarrow G \text{ is close to } \begin{array}{cc}
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\end{array} \text{ or } \begin{array}{cc}
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\end{array}"

- $x \sim y$ if $x \not\sim y$

(A) $\begin{array}{cc}
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\end{array}
\end{array}
\end{array}
\end{array}
\end{array} = 0$

(B) All are .

- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size $\frac{1}{3} n \pm 16\varepsilon n$ by (1).
"\( + \) is close to \( \frac{1}{27} \) \( \Rightarrow \) \( G \) is close to \( \) or \( \)"

- \( x \sim y \) if \( x \)
- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size \( \frac{1}{3} n \pm 16\varepsilon n \) by (1).
- The classes have the same color
“\( + \) is close to \( \frac{1}{27} \) \( \Rightarrow \) \( G \) is close to \( \) or \( \)"
"\(\Box + \Box\) is close to \(\frac{1}{27}\) \(\Rightarrow\) \(G\) is close to \(\triangle\) or \(\triangle\)"

- \(x \sim y\) if \(x \sim y\)
- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size \(\frac{1}{3}n \pm 16\varepsilon n\) by (1).
- The classes have the same color
“\(\square + \square\) is close to \(\frac{1}{27}\) \(\Rightarrow\) \(G\) is close to \(\square\) or \(\square\)”

- \(x \sim y\) if \(x \sim y\)

- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size \(\frac{1}{3}n \pm 16\varepsilon n\) by (1).
- The classes have the same color

\[\geq \frac{1}{2}\]
\textit{"A square plus a triangle is close to } \frac{1}{27} \Rightarrow \text{G is close to a triangle or a square."}

- \(x \sim y\) if \(x \neq y\)

- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size \(\frac{1}{3}n \pm 16\epsilon n\) by (1).
- The classes have the same color

Exact result: By recoloring edges.
Other permutations - maximizing 1342 and 2413

\[ 0.19657 \leq \sigma(1342) \leq \frac{2}{9} = 0.22222 \ldots \quad \text{AAHHS} \]
\[ \sigma(1342) \leq 0.1988373 \quad \text{BHLPUV} \]

\[ \frac{51}{511} = 0.0998 \ldots \leq \sigma(2413) \leq \frac{2}{9} = 0.22222 \quad \text{AAHHS} \]
\[ 0.1024732 \leq \sigma(2413) \quad \text{P} \]
\[ 0.10472 \ldots \leq \sigma(2413) \quad \text{PS} \]
\[ \sigma(2413) \leq 0.1047805 \quad \text{BHLPUV} \]

AAHHS ... Albert, Atkinson, Handley, Holton, Stromquist 2002
P...Presutti 2008
PS...Presutti, Stromquist 2010
BHLPUV... us
Thank you for your attention!