A counterexample to a conjecture on facial unique-maximal colorings

Bernard Lidicky* Kacy Messerschmidt† Riste Škrekovski‡

August 31, 2017

Abstract

A facial unique-maximum coloring of a plane graph is a proper vertex coloring by natural numbers where on each face $\alpha$ the maximal color appears exactly once on the vertices of $\alpha$. Fabrici and Göring [4] proved that six colors are enough for any plane graph and conjectured that four colors suffice. This conjecture is a strengthening of the Four Color theorem. Wendland [6] later decreased the upper bound from six to five. In this note, we disprove the conjecture by giving an infinite family of counterexamples. Thus we conclude that facial unique-maximum chromatic number of the sphere is five.

Keywords: facial unique-maximum coloring, plane graph.

1 Introduction

We call a graph planar if it can be embedded in the plane without crossing edges and we call it plane if it is already embedded in this way. A coloring of a graph is an assignment of colors to vertices. A coloring is proper if adjacent vertices receive distinct colors. A proper coloring of a graph embedded on some surface, where colors are natural numbers and every face has a unique vertex colored with a maximal color, is called a facial unique-maximum coloring, or FUM-coloring for short. The minimum $k$ such that a graph $G$ has a FUM-coloring using the colors $\{1, 2, \ldots, k\}$ is called the facial unique-maximum chromatic number of $G$ and is denoted $\chi_{fum}(G)$.

The cornerstone of graph colorings is the Four Color Theorem stating that every planar graph can be properly colored using at most four colors [2]. Fabrici and Göring [4] proposed the following strengthening of the Four Color Theorem.

Conjecture 1 (Fabrici and Göring). If $G$ is a plane graph, then $\chi_{fum}(G) \leq 4$. 

*Department of Mathematics, Iowa State University, USA. E-Mail: lidicky@iastate.edu
†Department of Mathematics, Iowa State University, USA. E-Mail: kacymess@iastate.edu
‡Faculty of Information Studies, Novo mesto & University of Ljubljana, Faculty of Mathematics and Physics & University of Primorska, FAMNIT, Koper, Slovenia. E-Mail: skrekovski@gmail.com
When stating the conjecture, Fabrici and Göring [4] proved that $\chi_{\text{fum}}(G) \leq 6$ for every plane graph $G$. Promptly, this coloring was considered by others. Wendland [6] decreased the upper bound to 5 for all plane graphs. Andova, Lidický, Lužar, and Škrekovski [1] showed that 4 colors suffice for outerplanar graphs and for subcubic plane graphs. Wendland [6] also considered the list coloring version of the problem, where he was able to prove the upper bound 7 and conjectured that lists of size 5 are sufficient. Edge version of the problem was considered by Fabrici, Jendrol', and Vrbjarová [5]. For more results on facially constrained colorings, see a recent survey written by Czap and Jendrol' [3].

In this note we disprove Conjecture 1.

Proposition 1. There exists a plane graph $G$ with $\chi_{\text{fum}}(G) > 4$.

Proof. Let $G$ be the graph depicted in Figure 1. It consists of the induced graph $H$ on the vertex set $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$, $H'$ (an isomorphic copy of $H$), and the edge $a_4a_2'$ connecting them. Suppose for contradiction that $G$ has a FUM-coloring with the colors in $\{1, 2, 3, 4\}$. The color 4 is assigned to at most one vertex in the outer face of $G$, so by symmetry we may assume that $a_1, a_2, a_3,$ and $a_4$ have colors in $\{1, 2, 3\}$. Next we proceed only with $H$ to obtain the contradiction.

By symmetry, assume $b_4$ is the unique vertex in $H$ that (possibly) has color 4. Without loss of generality, we assume $a_1, b_1,$ and $a_2$ are colored by $x, y,$ and $z$, respectively, where $\{x, y, z\} = \{1, 2, 3\}$. This forces $b_2$ to be colored with $x$, $a_3$ to be colored with $y$, and $b_3$ to be colored with $z$. Since $a_4$ is adjacent to vertices with colors $x, y,$ and $z$, it must have color 4, a contradiction. \qed

The contradiction in Proposition 1 is produced from the property of $H$ that every coloring of $H$ by colors $\{1, 2, 3, 4\}$, where every interior face has a unique-maximum color, has a vertex in the outer face colored by 4. We can generalize the counterexample in Figure 1 by constructing an infinite family of graphs $H = \{H_k\}_{k \geq 1}$ that can take the place of $H$. We construct a graph $H_k$ on $6k + 2$ vertices by first embedding the cycle $b_1b_2\cdots b_{3k+1}$ inside the cycle $a_1a_2\cdots a_{3k+1}$. For $1 \leq i \leq 3k$, add edges $a_ib_i$ and $b_ia_{i+1}$, then add the edges $a_{3k+1}b_{3k+1}$ and $b_{3k+1}a_1$. By this definition, the graph $H$ is equivalent to $H_1$. See Figure 2(a) for an example of a generalization of the counterexample.
Figure 2: More counterexamples to Conjecture 1.

It is possible to construct more diverse counterexamples by embedding copies of members of $\mathcal{H}$ inside the faces of any 4-chromatic graph $G$ and adding an edge from each copy to some vertex on the face it belongs to. It suffices to embed the graphs from $\mathcal{H}$ into a set of faces $K$ such that in every 4-coloring of $G$, there is at least one face in $K$ incident with a vertex of $G$ colored by 4. An example of this with $G$ being $K_4$ is given in Figure 2(b).

We now introduce a variation of Conjecture 1 with maximum degree and connectivity conditions added.

**Conjecture 2.** If $G$ is a connected plane graph with maximum degree 4, then $\chi_{fum}(G) \leq 4$.

Notice that we constructed a counterexample of maximum degree five. Moreover, removing the edge $a_4a'_2$ from the graph in Figure 1 gives a disconnected graph with maximum degree 4 that does not have a FUM-coloring with colors in $\{1, 2, 3, 4\}$. Recall that Andova et al. [1] showed that maximum degree 3 suffices.

For a surface $\Sigma$, we define the facial unique-maximum chromatic number of $\Sigma$,

$$\chi_{fum}(\Sigma) = \max_{G \hookrightarrow \Sigma} \chi_{fum}(G),$$

as the maximum of $\chi_{fum}(G)$ over all graphs $G$ embedded into $\Sigma$. Our construction and the result of Wendland [6] implies that $\chi_{fum}(S_0) = 5$, where $S_0$ is the sphere. Our result motivates to study this invariant for graphs on other surfaces. It would be interesting to have a similar characterization to Heawood number for other surfaces of higher genus.

**Problem 1.** Determine $\chi_{fum}(\Sigma)$ for surfaces $\Sigma$ of higher genus.

**Acknowledgment.** The project has been supported by the bilateral cooperation between USA and Slovenia, project no. BI–US/17–18–013. R. Škrekovski was partially supported by the Slovenian Research Agency Program P1–0383. B. Lidický was partially supported by NSF grant DMS-1600390.

**References**


