Goodman’s bound and Moon-Moser

Instead of determining the maximum number of edges in a $K_{k+1}$-free graph we may ask how many copies of $K_{k+1}$ are in a graph with some fixed number of edges. Moon and Moser gave a strong answer to this question that will lead to another proof of Turán’s theorem.

As a warm-up, we start with an extension of Mantel’s theorem. Let $N_s$ be the number of copies of $K_s$ in $G$.

**Theorem 1** (Goodman bound). *For every $n$-vertex graph $G$ with $m$ edges holds*

$$N_3 \geq \frac{m(4m-n^2)}{3n}.\]

The bound is not always tight. Tight asymptotic solution was obtained by Razborov and more precise count is in [https://arxiv.org/pdf/1712.00633.pdf](https://arxiv.org/pdf/1712.00633.pdf).

1: Show that Goodman bound is tight for Turán’s graphs $T_k(\ell \cdot k)$.

**Solution:** Let $T_k(\ell \cdot k)$ be a Turán’s graph on $n$ vertices, i.e., $n = k\ell$. Thee vertices induce a triangle iff they are from three different parts, so $N_3 = \binom{k}{3}\ell^3$. On the other hand, $e = \binom{k}{2}\ell^2$, hence we get

$$\binom{k}{3}\ell^3 = N_3 = \frac{m(4m-n^2)}{3n} = \frac{\binom{k}{2}\ell^2(4\binom{k}{2}\ell^2 - (k\ell)^2)}{3k\ell}\]

2: Prove Goodman bound. Outline of the proof: For every edge $xy$, give a lower bound on the number of triangles containing $xy$ (use $d(x), d(y), n$). Use the bound in $\sum$ over edges and change the $\sum$ to sum over vertices. And then use Cauchy-Schwartz.

**Solution:** The number of triangles using edge $xy$ is at least $d(x) + d(y) - n$ (as this counts the number of common neighbors of $x$ and $y$). Summing over all edges counts each triangle three times, so the total number of triangles is at least

$$\frac{1}{3} \sum_{xy \in E(G)} (d(x) + d(y) - n) = \frac{1}{3} \left( \sum_{x \in V(G)} d(x)^2 - nm \right).\]

Applying Cauchy-Schwartz inequality gives the total number of triangles is at least

$$\frac{1}{3} \left( \frac{1}{n} \left( \sum_{x \in V(G)} d(x) \right)^2 - nm \right) = \frac{4m}{3n} \left( m - \frac{n^2}{4} \right).\]

Remixed from notes of Cory Palmer by Bernard Lidický
Theorem 2 (Moon-Moser theorem). Let $N_s$ be the number of copies of $K_s$ in $G$. Then

$$N_{s+1} \geq \frac{N_s}{s^2 - 1} \left( \frac{s^2 N_s}{N_{s-1}} - n \right).$$

Proof. Let $S$ be a copy of $K_s$ in $G$. Define $d(A)$ as the number of common neighbors of a set of vertices $A$. As a generalization of the Handshaking lemma we have

$$\sum_S d(S) = (s+1)N_{s+1}. \quad (1)$$

3: Why is $(1)$ valid?

Solution: Double count pairs $(S, x)$, where $S$ is a copy of $K_S$ and $x$ is a vertex adjacent to all of $S$.

To prove Moon-Moser, we will count triples $(S', x, y)$ such that $S'$ is a copy of $K_{s-1}$ and $x$ and $y$ are (not necessarily distinct) vertices each adjacent to all vertices of $S'$.

4: Count $(S', x, y)$ by first picking $S'$, use $d(S')$ for the calculation and use Cauchy-Schwarz and Handshaking $(1)$ to give a lower bound.

Solution: Fixing, $S'$, then $x$ and $y$ must be common neighbors of $S'$, thus the total number of desired triples is

$$\sum_{S'} d(S')^2 \geq \frac{1}{N_{s-1}} \left( \sum_{S'} d(S') \right)^2 = \frac{s^2 N_s^2}{N_{s-1}}.$$

Where the first inequality is by Cauchy-Schwarz and the equality uses $(1)$.

5: Count $(S', x, y)$ by first considering $xy$. Make two cases - where you distinguish if $xy$ is an edge or not (non-edge also works as $x = y$). For $xy$ begin and edge, use $N_{s+1}$. For non-edge, what is $S' \cup \{x\}$ and use some bounds of how many choices are for $y$. The sum it all up!

Solution: If $xy$ is an edge, then $S'$ is inside a $K_{s+1}$ and there are $(s+1)sN_{s+1}$ ways to form a desired triple. If $xy$ is not an edge (allowing $x = y$), then $S' \cup \{x\}$ forms a $K_s$ and $y$ is adjacent to all vertices but $x$. Fix $S = K_s$ and pick a vertex $y$ adjacent to all but one vertex of $S$ (so $y$ may also be inside of $S$), then we have a triple $(S', x, y)$. There are at most $n - d(S)$ such choices for $y$. Thus the total number of triples is at most (using $(1)$).

$$(s+1)sN_{s+1} + \sum_S (n - d(S)) = (s+1)sN_{s+1} + nN_s - (s+1)N_{s+1} = (s^2 - 1)N_{s+1} + nN_s.$$

6: Combine the two estimates solving for $N_{s+1}$, which proves the theorem.

Solution: Obvious.
Corollary 3. Let $G$ be a graph with $e(G) = (1 - \frac{1-1}{x})\frac{n^2}{2}$. If $N_s$ is the number of copies of $K_{s+1}$ in $G$, then

$$N_{s+1} \geq \left(1 - \frac{s}{x}\right) \frac{n}{s+1} \cdot N_s.$$

7: Prove the corollary by induction on $s$.

Solution:

Proof. Induction on $s$. For $s = 1$ the claim follows immediately as $N_2$ is the number of edges and $N_1$ is the number of vertices. Let $s > 1$ and assume the statement for smaller values. By the Moon-Moser theorem and the inductive hypothesis we have

$$N_{s+1} \geq \frac{N_s}{s^2-1} \left(\frac{s^2 N_s}{N_{s-1}} - n\right) \geq \frac{N_s}{s^2-1} \left(\frac{s^2(1 - \frac{s-1}{x})\frac{n}{s} N_{s-1}}{N_{s-1}} - n\right).$$

Simplifying the RHS yields the claim.

The weak version of Turán’s theorem follows easily.

8: Prove (weak) Turán’s theorem using the corollary.

Solution:

Seventh proof of (weak) Turán’s theorem. Let $G$ be a graph with $e(G) > (1 - \frac{1}{s})\frac{n^2}{2}$, and fix $x$ such that $e(G) = (1 - \frac{1}{x})\frac{n^2}{2}$ (note that $x > s$). We can repeatedly apply Corollary 3 to get that $N_3 > 0$, $N_4 > 0$, \ldots, $N_s > 0$, $N_{s+1} > 0$, i.e., $G$ contains a copy of $K_{s+1}$.

Another important corollary states that when $G$ exceeds the number of edges given by Turán’s theorem, then not only do we have a copy of $K_{s+1}$ but in fact we have many copies. This property is called supersaturation.

Corollary 4. Fix $\epsilon > 0$ and $s$, then there exists $c = c(\epsilon, s)$ such that any $n$-vertex graph $G$ with $(1 - \frac{1}{s} + \epsilon)\frac{n^2}{2}$ edges, has at least $c n^{s+1}$ copies of $K_{s+1}$.

Proof. Set $\frac{1}{x} = \frac{1}{s} - \epsilon$, thus $x > s$.

9: Repeatedly apply Corollary 3 and get a lower bound on $N_{s+1}$.

Solution: Repeatedly applying Corollary 3 gives the number of copies of $K_{s+1}$ is

$$N_{s+1} \geq N_1 \prod_{i=1}^{s} \left[\left(1 - \frac{i}{x}\right) \frac{n}{i+1}\right].$$

Simplifying gives

$$N_{s+1} \geq \left(\frac{n}{x}\right)^{s+1} \left(\frac{x}{s+1}\right) \geq \left(\frac{n}{x}\right)^{s+1} \left(\frac{x}{s+1}\right)^{s+1} = \left(\frac{n}{s+1}\right)^{s+1}.$$

Note that in the corollary above if we have that $\epsilon = 0$, then $s = x$ and the lower bound on $N_{s+1}$ is simply 0.