ZERO FORCING NUMBER, MAXIMUM NULLITY, AND PATH COVER NUMBER OF SUBDIVIDED GRAPHS[*]

MINERVA CATRAL[†], ANNA CEPEK[‡], LESLIE HOGBEN[§], MY HUYNH[¶], KIRILL LAZEBNIK[‖], TRAVIS PETERS[**], AND MICHAEL YOUNG[††]

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Abstract. The zero forcing number, maximum nullity and path cover number of a (simple, undirected) graph are parameters that are important in the study of minimum rank problems. We investigate the effects on these graph parameters when an edge is subdivided to obtain a so-called edge subdivision graph. An open question raised by Barrett et al. in "Minimum rank of edge subdivisions of graphs," *Electronic Journal of Linear Algebra* (2009) 18: 530–563, is answered in the negative, and we provide additional evidence for an affirmative answer to another open question in that paper. It is shown that there is an independent relationship between the change in maximum nullity and zero forcing number caused by subdividing an edge once. Bounds on the effect of a single edge subdivision on the path cover number are presented, conditions under which the path cover number is preserved are given, and it is shown that the path cover number and the zero forcing number of a complete subdivision graph need not be equal.

Keywords. zero forcing number, maximum nullity, minimum rank, path cover number, edge subdivision, matrix, multigraph, graph

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1. Introduction. Let F be any field. For a (simple, undirected) graph $G = (V, E)$ that has vertex set $V = \{1, \ldots, n\}$ and edge set E, $\mathcal{S}(F, G)$ is the set of all symmetric $n \times n$ matrices A with entries from F such that for any non-diagonal entry a_{ij} in A, $a_{ij} \neq 0$ if and only if $ij \in E$. The *minimum rank* of G is

$$\mathrm{mr}(F, G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(F, G)\},$$

and the *maximum nullity* of G is

$$\mathrm{M}(F, G) = \max\{\mathrm{null}\, A : A \in \mathcal{S}(F, G)\}.$$

Note that $\mathrm{mr}(F, G) + \mathrm{M}(F, G) = |G|$, where $|G|$ is the number of vertices in G. Thus the problem of finding the minimum rank of a given graph is equivalent to the problem of determining its maximum nullity.

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[†]Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH 45207, USA (catralm@xavier.edu).

[‡]Bethany Lutheran College, Mankato, MN 56001, USA (Anna.Cepek@blc.edu).

[§] Department of Mathematics, Iowa State University, Ames, IA 50011, USA (lhogben@iastate.edu) & American Institute of Mathematics, 360 Portage Ave, Palo Alto, CA 94306 (hogben@aimath.org).

[¶]Department of Mathematics, Arizona State University, Tempe, AZ 85287, USA (mth79@cornell.edu).

[‖]Department of Mathematics, SUNY Geneseo, NY 14454, USA (kylazebnik@gmail.com).

[**]Natural and Mathematical Science Division, Culver-Stockton College, Canton, MO 63435, USA (tpeters319@gmail.com).

[††]Department of Mathematics, Iowa State University, Ames, IA 50011, USA (myoung@iastate.edu).

We say that a graph $H = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. The subgraph H is called an *induced subgraph* if for each $x, y \in V', xy \in E'$ if and only if $xy \in E$. Denote by $G[X]$ the induced subgraph of G with vertex set $X \subseteq V$; $G - W$ is used to denote $G[V \setminus W]$. The graph $G - \{v\}$ is also denoted by $G - v$. A graph G is the *union* of graphs $G_i = (V_i, E_i)$, $1 \leq i \leq h$, if $G = (\cup_{i=1}^{h} V_i, \cup_{i=1}^{h} E_i)$. A vertex v of a connected graph G is a *cut-vertex* if $G - v$ is disconnected. An edge e of a connected graph G is a *cut-edge* if $G - e$ is disconnected. The *rank spread* of G is $\mathrm{r}_v(F, G) = \mathrm{mr}(F, G) - \mathrm{mr}(F, G - v)$. One technique in computing minimum rank is by *cut-vertex reduction* (see, e.g., [6]), which is as follows: Suppose that v is a cut-vertex of G. For $i = 1, \dots, h$, let $W_i \subseteq V(G)$ be the vertices of the ith component of $G - v$ and let $G_i = G[\{v\} \cup W_i]$. Then $\mathrm{mr}(F, G) = \sum_{i=1}^{h} \mathrm{mr}(F, G_i - v) + \min\{2, \sum_{i=1}^{h} \mathrm{r}_v(F, G_i)\}$. For a graph $G = (V, E)$, the *degree* of $v \in V$, denoted $\deg v$, is the number of vertices in V that share an edge with v. A *leaf* vertex is a vertex of degree one. A *high degree* vertex is a vertex of degree greater than or equal to 3.

Observation 1.1. *Let G be a graph, let v be a leaf vertex of a graph G, and let F be a field. It is easy to see that* $\mathrm{mr}(F, G) - \mathrm{mr}(F, G - v) \leq 1$, *or equivalently,* $\mathrm{M}(F, G) \geq \mathrm{M}(F, G - v)$.

We consider two graph parameters that are related to the maximum nullity, namely the zero forcing number and the path cover number. The zero forcing number of a graph is the minimum number of black vertices initially needed to color all vertices black according to the color-change rule. The *color-change rule* is defined as follows: if G is a graph with each vertex colored either white or black, u is a black vertex of G and exactly one neighbor v of u is white, then change the color of v to be black. Let S be a subset of V. The *derived coloring of S* is the result of coloring every vertex in S black and every vertex not in S white, and then applying the color-change rule until no more changes are possible. A *zero forcing set* of G is a set $Z \subseteq V$ such that every vertex in the derived coloring of Z is black. The *zero forcing number* of G is

$$\mathrm{Z}(G) = \min\{|Z| : Z \text{ is a zero forcing set of } G\}.$$

A zero forcing set of G, Z, is called a *minimum zero forcing set* of G if $|Z| = \mathrm{Z}(G)$.

A *path* in G is a subgraph $H = (V', E')$ where $V' = \{u_1, \dots, u_k\}$ and $E' = \{u_1u_2, u_2u_3, \dots, u_{k-1}u_k\}$; a path is *even* or *odd* according as its number of vertices is even or odd. A *Hamiltonian path* of a graph G is a path that includes all the vertices of G. A *path cover* of G is a set of vertex disjoint paths, each of which is an induced subgraph of G, that contains all vertices of G. The *path cover number* of G is

$$\mathrm{P}(G) = \min\{|\mathcal{P}| : \mathcal{P} \text{ is a path cover of } G\}.$$

A path cover of G, $\mathcal{P}$, is called a *minimum path cover* of G if $|\mathcal{P}| = \mathrm{P}(G)$.

The relationships between $\mathrm{M}(F, G), \mathrm{Z}(G)$ and $\mathrm{P}(G)$ for any graph G are discussed in papers devoted to the study of minimum rank problems. For extensive surveys on minimum rank and related problems, see [6] or [7].

Theorem 1.2. [1] *For any graph G,* $\mathrm{M}(F, G) \leq \mathrm{Z}(G)$.

Theorem 1.3. [8] *For any graph G,* $\mathrm{P}(G) \leq \mathrm{Z}(G)$.

In [2], examples of graphs are given to show that both $\mathrm{M}(F, G) < \mathrm{P}(G)$ and $\mathrm{P}(G) < \mathrm{M}(F, G)$

are possible. In particular, $\mathrm{M}(F,G) < \mathrm{Z}(G)$ is possible. However, all three parameters give equality for graphs that are trees.

THEOREM 1.4. [1, 5, 9] *For any tree T, $\mathrm{M}(F,T) = \mathrm{P}(T) = \mathrm{Z}(T)$.*

Following the notation in [3], we give the following definitions. Let $e = uv$ be an edge of G. Define G_e to be the graph obtained from G by inserting a new vertex w into V, deleting the edge e and inserting edges uw and wv. We say that that the edge e has been *subdivided* and call G_e an *edge subdivision* of G. A *complete subdivision graph* $\widehat{G}$ is obtained from a graph G by subdividing every edge of G once. In [3] and [10], the authors investigate the maximum nullity and zero forcing number of graphs obtained by a finite number of edge subdivisions of a given graph and, among other results, establish the following two propositions about the effect of an edge subdivision on the zero forcing number and maximum nullity.

PROPOSITION 1.5. [3, 10] *Let G be a graph and let e be an edge of G. Then*

$$\mathrm{M}(F,G) \leq \mathrm{M}(F,G_e) \leq \mathrm{M}(F,G)+1 \quad \textit{and} \quad \mathrm{Z}(G) \leq \mathrm{Z}(G_e) \leq \mathrm{Z}(G)+1.$$

PROPOSITION 1.6. [3, 10] *Let G be a graph and let e be an edge of G incident to a vertex of degree at most 2. If $F \neq \mathbb{Z}_2$, then $\mathrm{M}(F,G) = \mathrm{M}(F,G_e)$ and $\mathrm{Z}(G) = \mathrm{Z}(G_e)$.*

The paper [3] concludes with a list of open questions, including the following two questions.

QUESTION 1.7. *Let F be a field. Suppose G is a graph in which each vertex has degree at least 3 and H is a graph that has one less edge subdivision than $\widehat{G}$. Is it always the case that $\mathrm{M}(F,H) < \mathrm{M}(F,\widehat{G})$?*

QUESTION 1.8. *Is $\mathrm{M}(F,\widehat{G}) = \mathrm{Z}(\widehat{G})$ for every field F and graph G?*

In [3], the authors provide the following substantial result toward an affirmative answer to Question 1.8.

THEOREM 1.9. [3] *If $G = (V,E)$ has a Hamiltonian path then $\mathrm{M}(F,\widehat{G}) = \mathrm{Z}(\widehat{G}) = m - n + 2$ and $\mathrm{mr}(F,\widehat{G}) = 2n - 2$, where $n = |V|$ and $m = |E|$.*

In Section 2 we provide additional evidence of an affirmative answer to Question 1.8, including establishing that $\mathrm{M}(F,\widehat{G}) = \mathrm{Z}(\widehat{G})$ if G does not have a cut-edge. In Section 3 we give an example that provides a negative answer to Question 1.7. We also present examples showing that there is an independent relationship between the change in maximum nullity and zero forcing number caused by a single edge subdivision in a graph G . In Section 4, we give bounds on the effect of a single edge subdivision on the path cover number and give conditions under which the path cover number is preserved. We also provide an example to show that $\mathrm{P}(\widehat{G})$ need not equal $\mathrm{Z}(\widehat{G})$ for an arbitrary graph G.

2. Complete edge subdivision graphs. In [3] it was shown that $\mathrm{M}(F,\widehat{G}) = \mathrm{Z}(\widehat{G})$ if G has a Hamiltonian path. In this section we establish $\mathrm{M}(F,\widehat{G}) = \mathrm{Z}(\widehat{G})$ for other conditions on G, specifically for graphs G such that G is a cactus or has no cut-edge.

A *cactus* is a graph in which any two cycles share at most one vertex. We use Row's work on cacti to show that the zero forcing number and maximum nullity of a complete subdivision of any cactus is equal.

Proposition 2.1. [11] *Let G be a cactus in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors. Then* $\mathrm{M}(\mathbb{R}, G) = \mathrm{Z}(G)$.

Proposition 2.2. *If $G = (V, E)$ is a cactus, then* $\mathrm{M}(F, \hat{\overline{G}}) = \mathrm{Z}(\hat{\overline{G}})$.

Proof. Let $G = (V, E)$ be a cactus. We perform a complete subdivision on G. Notice then that $\hat{\overline{G}}$ is a cactus. Furthermore, each cycle in $\hat{\overline{G}}$ is even (and has a vertex of degree two). Thus $\mathrm{M}(\mathbb{R}, \hat{\overline{G}}) = \mathrm{Z}(\hat{\overline{G}})$. If H is a cycle or tree, then $\mathrm{M}(F, H) = \mathrm{M}(\mathbb{R}, H)$. Since cut-vertex reduction preserves field independence (see [6]), $\mathrm{M}(F, \hat{\overline{G}}) = \mathrm{Z}(\hat{\overline{G}})$ for every cactus G. □

To prove that $\mathrm{M}(F, \hat{\overline{G}}) = \mathrm{Z}(\hat{\overline{G}})$ for every G that does not have a cut-edge, we first generalize the set of complete edge subdivision graphs.

Definition 2.3. Let $\mathcal{K}$ be the family of bipartite graphs $G = (V(G), E(G))$ such that there is a bipartition of the vertices $V(G) = X \,\dot\cup\, Y$ with $\deg x \leq 2$ for all $x \in X$.

Note that every path is in $\mathcal{K}$, and every even cycle is in $\mathcal{K}$. An odd cycle is not bipartite, so it is not in $\mathcal{K}$. If G is any connected bipartite graph, then the (unordered) pair of bipartition sets is uniquely determined. If $G \in \mathcal{K}$ and G has a high degree vertex, then the bipartition sets X and Y such that $V(G) = X \,\dot\cup\, Y$ and $\deg x \leq 2$ for all $x \in X$ are uniquely determined. When the sets X, Y such that $V(G) = X \,\dot\cup\, Y$ and $\deg x \leq 2$ for all $x \in X$ are not uniquely determined, we often make a choice, possibly subject to some additional condition(s). When X and Y are specified by uniqueness or by choice, we write $X(G)$ for X and $Y(G)$ for Y.

Proposition 2.4. *A graph H is a complete subdivision graph of some graph G if and only if $H \in \mathcal{K}$, H does not contain a cycle on four vertices, and* $\deg x = 2$ *for every* $x \in X(H)$.

Proof. The forward direction is clear. For the converse, we reconstruct G from H. It is sufficient to do so for a connected graph, and then take the union of connected components, so assume H is connected. If H has no high degree vertex, then H is an even cycle or odd path (an even path is not allowed because one vertex in each bipartition set of such a path has degree one), and thus H is a complete subdivision graph. So assume H has a high degree vertex. For each $x \in X(H)$ with neighbors $y_1, y_2 \in Y(H)$, delete edges xy_1 and xy_2 and vertex x and add edge y_1y_2. This method creates a graph G such that $H = \hat{\overline{G}}$: G is a graph, since no duplicate edges are created (two vertices $x_1, x_2 \in X$ with the same neighbors $y_1, y_2 \in Y(G)$ would have created a cycle on four vertices in H, which we expressly disallow). □

Conjecture 2.5. *If $G \in \mathcal{K}$, then* $\mathrm{M}(F, G) = \mathrm{Z}(G)$.

By Proposition 2.4, every complete subdivision graph is in $\mathcal{K}$, so this conjecture generalizes a conjecture that $\mathrm{M}(F, \hat{\overline{G}}) = \mathrm{Z}(\hat{\overline{G}})$ for all graphs G.

The method by which we show $\mathrm{M}(F, \hat{\overline{G}}) = \mathrm{Z}(\hat{\overline{G}})$ for graphs without a cut-edge requires knowing that certain diagonal entries of a matrix are zero. A graph $G \in \mathcal{K}$ is *special* if there exists a matrix $A \in \mathcal{S}(G)$ such that

1. $\operatorname{null} A = \mathrm{M}(F, G)$.
2. If $x \in X(G)$, then $a_{xx} = 0$.

For a special graph G, a matrix $A \in \mathcal{S}(G)$ satisfying conditions (1) and (2) is *optimal* for G.

Let G be a graph and let $C = (V_C, E_C)$ be a cycle that is a subgraph of G. A *subdivided chordal path* of G is a path $P = (v_1, \dots, v_{2k+1})$ in G such that $v_1, v_{2k+1} \in V_C$, $\deg_G v_i = 2$ for $i = 2, 3, \dots, 2k$, and $v_i \notin V_C$ for $i = 2, 3, \dots, 2k$.

THEOREM 2.6. *Let G' be a graph in $\mathcal{K}$ and let G be obtained from G' by removing a subdivided chordal path $P = (v_1, v_2, v_3)$ of G' between two vertices in $V(G)$. If $\mathrm{M}(F, G) = \mathrm{Z}(G)$ and G is special, then $\mathrm{M}(F, G') = \mathrm{Z}(G')$ and G' is special.*

Proof. Suppose that $\mathrm{M}(F, G) = \mathrm{Z}(G)$ and G is special. Let $Q = (v_1, u_2, \dots, u_{2k}, v_3)$ be another path that connects v_1 and v_3. Since $G' \in \mathcal{K}$ and $v_1, v_3 \in Y(G')$, $\deg_G u_{2i} = \deg_{G'} u_{2i} = 2$ for $i = 1, \dots, k$. Let A be an optimal matrix for G, so the diagonal entries of A in the column vectors $\mathbf{a}_{u_{2i}}$ associated with vertices $u_{2i}, i = 1, \dots, k$ are all zero. Since the only vertices adjacent to u_2 are v_1 and u_3, $\mathbf{a}_{u_2}$ has nonzero entries exactly in rows v_1 and u_3, and similarly, $\mathbf{a}_{u_4}$ has nonzero entries exactly in rows u_3 and u_5. We can take a linear combination of these two vectors to cancel the nonzero entry in row u_3, to obtain a column vector with nonzero entries exactly in rows v_1, u_5. We iterate this process with column vectors to obtain a column vector $\mathbf{c}$ with non-zero entries in exactly rows v_1, v_3. Let $A' = [a'_{ij}]$ be A with the extra column $\mathbf{c}$ and extra row $\mathbf{c}^T$ and zero as the new diagonal entry. We know $A' \in \mathcal{S}(G')$. Since G is an induced subgraph of G', $\mathrm{mr}(F, G) \le \mathrm{mr}(F, G')$. Since $\mathrm{rank}(A') = \mathrm{rank}(A)$, $\mathrm{mr}(F, G) = \mathrm{mr}(F, G')$. Hence, $\mathrm{M}(F, G') = \mathrm{M}(F, G) + 1$.

Since $a'_{xx} = 0$ for every $x \in X(G')$, G' is special. Note that $\mathrm{Z}(G) + 1 = \mathrm{M}(F, G) + 1 = \mathrm{M}(F, G') \le \mathrm{Z}(G') \le \mathrm{Z}(G) + 1$. Hence, $\mathrm{Z}(G') = \mathrm{M}(F, G')$. □

Although this paper is primarily concerned with simple graphs, multigraphs are a useful tool. A *multigraph* $G = (V, E)$ is a general graph in which E is a multiset of two-element subsets of vertices. That is, a multigraph allows multiple copies of an edge vw (where $v \ne w$), but a loop vv is not permitted. For a field $F \ne \mathbb{Z}_2$, the *maximum nullity of a multigraph G* of order n over F, denoted by $\mathrm{M}(F, G)$, is the largest possible nullity over all matrices $A \in F^{n \times n}$ whose ijth entry a_{ij} (for $i \ne j$) is zero if i and j are not adjacent in G, is nonzero if ij is a single edge, and is any element of F if ij is a multiple edge. In the case that $F = \mathbb{Z}_2$ and ij is a multiple edge, a_{ij} is 0 if the number of copies of edge ij is even and 1 if it is odd. If a multigraph does not have any multiple edges then it is a (simple) graph. Observe that if G is a multigraph, then $\overset{\circ}{\overrightarrow{G}}$ is a (simple) graph and $\overset{\circ}{\overrightarrow{G}} \in \mathcal{K}$.

The *contraction* of edge $e = uv$ of G is the multigraph obtained from G by identifying the vertices u and v, deleting any loops that arise in this process. A set $R \subset V(G)$ is a *separating set* of a graph G if $G - R$ has more connected components than G does; in this case R is called an r-separating set where $r = |R|$. A 1-separating set is a cut-vertex, and cut-vertex reduction is a standard technique for computing minimum rank/maximum nullity. Van der Holst has established a 2-separating set reduction for computing maximum nullity using multigraphs. A 2-*separation* of G is a pair of subgraphs $(G_1(R), G_2(R))$ such that $V(G_1(R)) \cap V(G_2(R)) = R = \{r_1, r_2\}$, $V(G_1(R)) \cup V(G_2(R)) = V(G)$, $E(G_1(R)) \cap E(G_2(R)) = \emptyset$, and $E(G_1(R)) \cup E(G_2(R)) = E(G)$. We introduce some notation for the multigraphs needed for van der Holst's 2-separation theorem. For $i = 1, 2$, $H_i(R)$ is the graph or multigraph obtained from $G_i(R)$ by adding edge $r_1 r_2$. If $r_1 r_2 \notin E(G_i(R))$, $H_i(R)$ is a (simple) graph; otherwise $H_i(R)$ is a multigraph having two edges between r_1 and r_2 (with every other pair of vertices either nonadjacent or joined by exactly one edge). At most one of $H_1(R), H_2(R)$ has a multiple edge. For $i = 1, 2$, $\widehat{G}_i(R)$ is the multigraph obtained from $H_i(R)$ by contracting an edge r_1r_2 (note that van der Holst uses the notation $\overline{G}_i(R)$ for what we denote by $\widehat{G}_i(R)$, but $\overline{G}_i(R)$ may cause confusion with a complement).

THEOREM 2.7. [12] *Let G be a (simple) graph, let $(G_1(R), G_2(R))$ be a 2-separation of G. Then*

$$\mathrm{M}(F, G) = \max \left\{ \begin{array}{c} \mathrm{M}(F, G_1(R)) + \mathrm{M}(F, G_2(R)), \\ \mathrm{M}(F, H_1(R)) + \mathrm{M}(F, H_2(R)), \\ \mathrm{M}(F, \widehat{G}_1(R)) + \mathrm{M}(F, \widehat{G}_2(R)), \\ \mathrm{M}(F, G_1(R) - r_1) + \mathrm{M}(F, G_2(R) - r_1), \\ \mathrm{M}(F, G_1(R) - r_2) + \mathrm{M}(F, G_2(R) - r_2), \\ \mathrm{M}(F, G_1(R) - R) + \mathrm{M}(F, G_2(R) - R) \end{array} \right\} - 2.$$

LEMMA 2.8. *Let G be a graph in $\mathcal{K}$ and $(G_1(R), G_2(R))$ be a 2-separation of G. If $G_1(R)$ is an even path with endpoints r_1 and r_2 and $r_1r_2 \notin E(G)$, then $\mathrm{M}(F, G) = \mathrm{M}(F, H_1(R)) + \mathrm{M}(F, H_2(R)) - 2$ (or equivalently, $\mathrm{mr}(F, G) = \mathrm{mr}(F, H_1(R)) + \mathrm{mr}(F, H_2(R))$) and $H_1(R), H_2(R) \in \mathcal{K}$.*

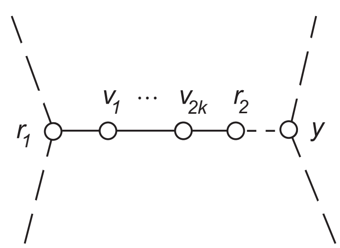


Fig. 2.1: Illustration for Lemma 2.8

Proof. Let $G_i = G_i(R), H_i = H_i(R), \widehat{G}_i = \widehat{G}_i(R), i = 1, 2$. Since $r_1r_2 \notin E(G)$, H_1 and H_2 are (simple) graphs, and it is clear that $H_1, H_2 \in \mathcal{K}$. To show $\mathrm{M}(F, G) = \mathrm{M}(F, H_1) + \mathrm{M}(F, H_2) - 2$, by Theorem 2.7 it suffices to prove the following inequalities.

- $\mathrm{M}(F, H_1) + \mathrm{M}(F, H_2) \geq \mathrm{M}(F, G_1) + \mathrm{M}(F, G_2)$: Since G_1 is a path and H_1 is a cycle, $\mathrm{M}(F, G_1) = \mathrm{M}(F, H_1) - 1$. Since G_2 is obtained from H_2 by deleting the edge r_1r_2, $\mathrm{M}(F, H_2) \geq \mathrm{M}(F, G_2) - 1$. Hence,

$$\begin{aligned} \mathrm{M}(F, H_1) + \mathrm{M}(F, H_2) &\geq \mathrm{M}(F, G_1) + 1 + \mathrm{M}(F, G_2) - 1 \\ &= \mathrm{M}(F, G_1) + \mathrm{M}(F, G_2). \end{aligned}$$

- $\mathrm{M}(F, H_1) + \mathrm{M}(F, H_2) \geq \mathrm{M}(F, \widehat{G}_1) + \mathrm{M}(F, \widehat{G}_2)$: Since $\widehat{G}_1$ is a cycle, $\mathrm{M}(F, \widehat{G}_1) = 2 = \mathrm{M}(F, H_1)$. If $\deg r_2 = 1$, then r_2 is a leaf of H_2, so by Observation 1.1, $\mathrm{M}(F, H_2) \geq \mathrm{M}(F, H_2 - r_2) = \mathrm{M}(F, \widehat{G}_2)$. So assume $\deg r_2 = 2$ and let $r_2y \in E(G)$ and $y \neq v_{2k}$. Note that $r_1y \notin E(G)$ since r_1, y are in the same bipartition set and $r_1 \neq y$. Observe that $H_2 = (\widehat{G}_2)_e$ where $e = r_2y$. By Proposition 1.5, $\mathrm{M}(F, \widehat{G}_2) \leq \mathrm{M}(F, H_2)$, and the desired inequality follows.

- For $i = 1, 2$, $\mathrm{M}(F, H_1)+\mathrm{M}(F, H_2) \geq \mathrm{M}(F, G_1 - r_i)+\mathrm{M}(F, G_2 - r_i)$: Observe that $\mathrm{M}(F, G_1 - r_i) = 1 = \mathrm{M}(F, H_1) - 1$. Since $G_2 - r_i = H_2 - r_i$, $\mathrm{M}(F, H_2) \geq \mathrm{M}(F, H_2 - r_i) - 1 = \mathrm{M}(F, G_2 - r_i) - 1$, and the desired inequality follows.
- $\mathrm{M}(F, H_1) + \mathrm{M}(F, H_2) \geq \mathrm{M}(F, G_1 - R) + \mathrm{M}(F, G_2 - R)$: Observe that $\mathrm{M}(F, G_1 - R) = 1 = \mathrm{M}(F, H_1) - 1$. Since $G_2 - r_1 = H_2 - r_1$, $\mathrm{M}(F, H_2) \geq \mathrm{M}(F, H_2 - r_1) - 1 = \mathrm{M}(F, G_2 - r_1) - 1$. Since r_2 is a leaf vertex of $G_2 - r_1$, $\mathrm{M}(F, G_2 - R) \leq \mathrm{M}(F, G_2 - r_1)$, and thus $\mathrm{M}(F, H_2) \geq \mathrm{M}(F, G_2 - R) - 1$. Hence the desired inequality follows.

□

If $V(L) \subset V(G)$ and $A = [a_{uv}] \in \mathcal{S}(L)$, then the *embedding* $\tilde{A} = [\tilde{a}_{uv}]$ of A for G is the $|G| \times |G|$ matrix defined by $\tilde{a}_{uv} = a_{uv}$ if $u, v \in V(L)$ and 0 otherwise. A *decomposition* of a graph G is a pair of graphs (L_1, L_2) such that

1. $V(G) = V(L_1) \cup V(L_2)$.
2. $|V(L_1) \cap V(L_2)| = 2$.
3. $|E(L_1) \cap E(L_2)| = 0$ or 1.
4. $E(G) = (E(G_1) \cup E(G_2)) \setminus (E(G_1) \cap E(G_2))$.

Every 2-separation $(G_1(R), G_2(R))$ of G is a decomposition of G, but not conversely. A decomposition (L_1, L_2) of a graph $G \in \mathcal{K}$ is a *special decomposition* if it satisfies all of the following conditions:

1. $L_1, L_2 \in \mathcal{K}$.
2. $\mathrm{mr}(F, G) = \mathrm{mr}(F, L_1) + \mathrm{mr}(F, L_2)$. Equivalently, $\mathrm{M}(F, G) = \mathrm{M}(F, L_1) + \mathrm{M}(F, L_2) - 2$.
3. For $r \in V(L_1) \cap V(L_2)$, either $r \in Y(L_1) \cap Y(L_2)$ or $r \in X(L_1) \cap X(L_2)$.

Lemma 2.9. *Suppose (L_1, L_2) is a decomposition of a graph G. If $A_k \in \mathcal{S}(L_k), k = 1, 2$, then there exists $\alpha \in F$ such that $A = A_1 + \alpha A_2 \in \mathcal{S}(G)$. If $\mathrm{mr}(F, G) = \mathrm{mr}(F, L_1) + \mathrm{mr}(F, L_2)$ and $\operatorname{rank} A_k = \mathrm{mr}(F, L_k)$, for $k = 1, 2$, then $\operatorname{rank} A = \mathrm{mr}(F, G)$ (for this α). If (L_1, L_2) is a special decomposition of $G \in \mathcal{K}$ and L_1 and L_2 are special, then G is special.*

Proof. If $E(L_1) \cap E(L_2) = \emptyset$, choose $\alpha = 1$. If $E(L_1) \cap E(L_2) = \{zw\}$ choose $\alpha = -a^{(1)}_{zw}/a^{(2)}_{zw}$ where $A_k = [a^{(k)}_{ij}], k = 1, 2$. Then $A \in \mathcal{S}(G)$ and $\operatorname{rank} A \leq \operatorname{rank} A_1 + \operatorname{rank} A_2$, so $\mathrm{mr}(F, G) = \mathrm{mr}(F, L_1) + \mathrm{mr}(F, L_2)$ implies $\operatorname{rank} A = \mathrm{mr}(F, G)$.

Now suppose (L_1, L_2) is a special decomposition of G and L_1, L_2 are special. Construct $A = [a_{ij}]$ as previously using optimal A_k for $L_k, k = 1, 2$. We claim A is optimal for G and thus G is special. It is already established that $\operatorname{null} A = \mathrm{M}(F, G)$ and since for $r \in V(L_1) \cap V(L_2)$, either $r \in Y(L_1) \cap Y(L_2)$ or $r \in X(L_1) \cap X(L_2)$, the required zeros on the diagonal are preserved. □

Theorem 2.10. *Let G' be a graph in $\mathcal{K}$ and let G be obtained from G' by removing a subdivided chordal path $P = (v_1, \ldots, v_{2k+1})$ of G' between two vertices in $V(G)$. If $\mathrm{M}(F, G) = \mathrm{Z}(G)$ and G is special, then $\mathrm{M}(F, G') = \mathrm{Z}(G')$ and G' is special.*

Proof. Theorem 2.6 covers the case $k = 1$, so assume $k \geq 2$. Let $r_1 = v_1, r_2 = v_{2k}$, and $R = \{r_1, r_2\}$. Let $G_1(R) = (r_1, v_2, \ldots, v_{2k-1}, r_2)$ be a path in G' and $G_2(R) = G' - \{v_2, \ldots, v_{2k-1}\}$, so $(G_1(R), G_2(R))$ is a 2-separation of G'; see Figure 2.2. Since $r_1 r_2 \notin E(G')$, H_1 is a cycle on $2k$ vertices and H_2 is obtained from G by adding the subdivided chordal path (v_1, r_2, v_{2k+1}); see

Figure 2.2. By Theorem 2.6 H_2 is special, and by Lemma 2.8 $\mathrm{mr}(F,G') = \mathrm{mr}(F,H_1) + \mathrm{mr}(F,H_2)$. Thus (H_1,H_2) is a special decomposition of G', and so by Lemma 2.9, G' is special. Furthermore, we have

$$\begin{aligned} \mathrm{M}(F,G') &= \mathrm{M}(F,H_1) + \mathrm{M}(F,H_2) - 2 \\ &= \mathrm{M}(F,H_2) \\ &= \mathrm{Z}(H_2) \\ &= \mathrm{Z}(G') \end{aligned}$$

by subdividing edges incident to a vertex of degree two. □

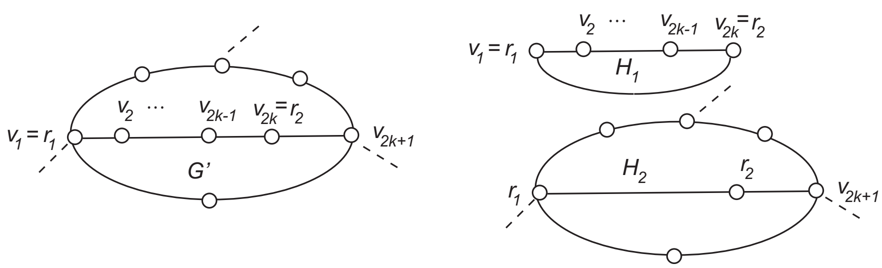


Fig. 2.2: Illustration for Theorem 2.10

Lemma 2.11. *Let G be a graph. If cycles C_1, C_2 of G intersect in $k > 1$ paths, then there is a cycle C_3 of G such that C_1 and C_3 intersect in exactly one path and that path has at least two vertices.*

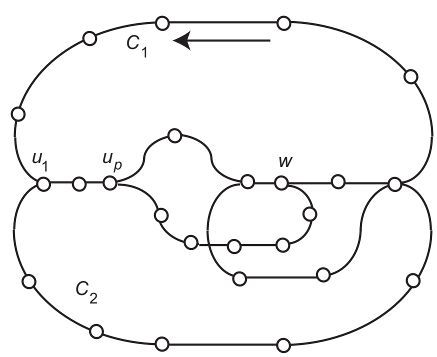
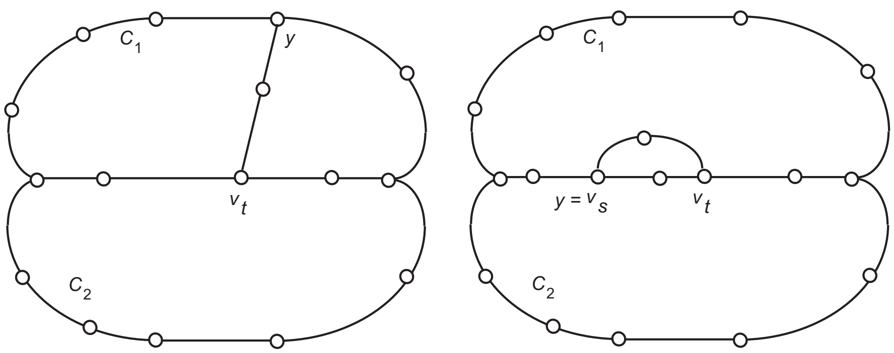


Fig. 2.3: Illustration for Lemma 2.11

Proof. Choose an orientation for C_1. With this orientation, each vertex $v \in C_1$ has a predecessor and a successor. Let $P = (u_1, \ldots, u_p)$ be a path in $C_1 \cap C_2$ that conforms to the orientation and that is maximal in the sense that the predecessor of u_1 in C_1 is not in C_2 and the successor of u_p in C_1 is not in C_2. Impose the orientation of P on C_2. Let w be the first vertex in C_2 after u_p that is also in C_1 (see Figure 2.3). Let P_i be the path in C_i connecting u_p and w (following the orientation of C_i). Define C_3 to be the cycle enclosed by P_1 and P_2. Then C_1 intersects C_3 in exactly P_1, and $u_p, w \in V(P_1)$. □

Lemma 2.12. *Let G be a graph in $\mathcal{K}$. Suppose cycles C_1, C_2 of G intersect in exactly one path P and none of the interior vertices of P is a cut-vertex. Then G contains a subdivided chordal path of some cycle.*

Proof. Let $P = (v_1, \ldots, v_m)$. The proof is by strong induction on the number ℓ of high degree vertices among the interior vertices $v_i, i = 2, \ldots, m-1$. If $\ell = 0$, then P is a subdivided chordal path of G. So assume that if two cycles of G intersect in exactly one path that has $q < \ell$ high degree interior vertices, then G contains a subdivided chordal path, and suppose P has ℓ high degree interior vertices. Let v_t be a high degree interior vertex. Since v_t is not a cut-vertex, there exists a path Q_1 that connects v_t to some other vertex $y \in V(C_1)$ (if necessary reverse the names of C_1 and C_2) and such that $V(Q) \cap V(C_1) = \{v_t, y\}$. We consider two cases depending on whether or not y is on P, as illustrated in Figure 2.4.



Case 1: y not on P Case 2: y on P

Fig. 2.4: Illustration for Lemma 2.12

Case 1. $y \notin V(P)$: Let Q_2 be the path in C_1 between y and v_t that does not contain v_m. Then $(v_1, v_2, \ldots, v_t)$, Q_1, and Q_2 form a cycle C_3 that intersects C_2 in path $P' = (v_1, v_2, \ldots, v_t)$. Since P' has fewer high degree interior vertices, G contains a subdivided chordal path.

Case 2. $y \in V(P)$: Let P' be the subpath of P between $v_s = y$ and v_t, so P' and Q_1 form a cycle C_3 that intersects C_2 in path $P' = (v_s, \ldots, v_t)$. Since P' has fewer high degree interior vertices, G contains a subdivided chordal path. □

Proposition 2.13. *Suppose G has a cut-vertex v. For $i = 1, \ldots, h$, let $W_i \subseteq V(G)$ be the vertices of the ith component of $G - v$ and let G_i be the subgraph induced by $\{v\} \cup W_i$. If $\mathrm{r}_v(F, G_1) = 0$, then*

$$\mathrm{mr}(F, G) = \mathrm{mr}(F, G_1) + \mathrm{mr}(F, G - W_1).$$

Proof. By cut-vertex reduction $\mathrm{mr}(F, G) = \sum_{i=1}^{h} \mathrm{mr}(F, G_i - v) + \min\{2, \sum_{i=1}^{k} \mathrm{r}_v(F, G_i)\}$. Since $r_v(F, G_1) = 0$, $\mathrm{mr}(F, G) = \mathrm{mr}(F, G_1 - v) + \sum_{i=2}^{k} \mathrm{mr}(F, G_i - v) + \min\{2, \sum_{i=2}^{k} \mathrm{r}_v(F, G_i)\} = \mathrm{mr}(F, G_1) + \mathrm{mr}(F, G - W_1)$. □

Proposition 2.14. *Let $G = (V, E)$ be a graph containing a cycle C on $k \geq 3$ vertices that contains exactly one high degree vertex, v. Then* $\mathrm{mr}(F, G) = \mathrm{mr}(F, C) + \mathrm{mr}(F, G - V(C - v))$*, or equivalently,* $\mathrm{M}(F, G) = \mathrm{M}(F, G - V(C - v)) + 1$*. Furthermore,* $\mathrm{Z}(G) \leq \mathrm{Z}(G - V(C - v)) + 1$*. If* $\mathrm{M}(F, G - V(C - v)) = \mathrm{Z}(G - V(C - v))$*, then* $\mathrm{M}(F, G) = \mathrm{Z}(G)$*.*

Proof. From Proposition 2.13, $\mathrm{mr}(F, G) = \mathrm{mr}(F, C) + \mathrm{mr}(F, G - V(C - v))$, so

$$|G| - \mathrm{M}(F, G) = (k - 2) + |G| - (k - 1) - \mathrm{M}(F, G - V(C - v)),$$

or $\mathrm{M}(F, G) = \mathrm{M}(F, G - V(C - v)) + 1$. To establish $\mathrm{Z}(G) \leq \mathrm{Z}(G - V(C - v)) + 1$, we exhibit a zero forcing set of order $\mathrm{Z}(G - V(C - v)) + 1$. Let B be a minimum zero forcing set for $G - V(C - v)$, and let x be a neighbor of v in C. Then $B \cup \{x\}$ is a zero forcing set for G. If $\mathrm{M}(F, G - V(C - v)) = \mathrm{Z}(G - V(C - v))$, then $\mathrm{Z}(G - V(C - v)) + 1 = \mathrm{M}(F, G - V(C - v)) + 1 = \mathrm{M}(F, G) \leq \mathrm{Z}(G) \leq \mathrm{Z}(G - V(C - v)) + 1$ so we have equality throughout. □

Remark 2.15. Every cycle on an even number of vertices is special. Specifically, for a cycle C on $2k$ vertices, the adjacency matrix is optimal if k is even, and if k is odd, an optimal matrix is $A = [a_{ij}] \in \mathcal{S}(F, C)$ where $a_{i,i+1} = 1, i = 1, \ldots, 2k - 1$ and $a_{1,2k} = -1$ (this is valid over every field F).

Theorem 2.16. *If G is a graph in $\mathcal{K}$ that does not have a cut-edge, then G is special and* $\mathrm{M}(F, G) = \mathrm{Z}(G)$*.*

Proof. We prove the following two statements by induction on the number of cycles for a connected graph $G \in \mathcal{K}$ that does not have a cut-edge.

(A) G is a cycle or G contains a cycle with exactly one high degree vertex or G has a subdivided chordal path.

(B) G is special and $\mathrm{M}(F, G) = \mathrm{Z}(G)$.

Both (A) and (B) are clear for all cycles in $\mathcal{K}$, and thus for all connected graphs $G \in \mathcal{K}$ such that G has no cut edge and at most one cycle. Assume both (A) and (B) are true for all connected graphs G having no cut-edge and at most $k \geq 1$ cycles. Let G' be a connected graph in $\mathcal{K}$ that does not have a cut-edge and has $k + 1$ cycles.

Case 1. G' has a cut-vertex: If G' has a cycle with exactly one high degree vertex, then (A) is true and (B) follows from Proposition 2.14 and the induction hypothesis. If G' does not have a cycle with exactly one high degree vertex, then consider the blocks $G_1, \ldots G_b$ of G'. Since G' has a cut-vertex and no cut-edge, $b > 1$ and each block contains a cycle. Thus G_1 has fewer than $k+1$ cycles. Since G' does not contain a cycle with exactly one high degree vertex, G_1 is not a cycle and does not contain a cycle with at most one high degree vertex. By the induction hypothesis, G_1 contains a subdivided chordal path. Since G_1 is a block of G', G' contains a subdivided chordal path. Thus (A) is true, and (B) follows from Theorem 2.10 and the induction hypothesis.

Case 2. G' does not have a cut-vertex: Since G' has more than one cycle and G' does not have a cut-vertex, G' has two cycles that intersect in one path on at least two vertices or that intersect in more than one path. Then by Lemma 2.11, G' has two cycles that intersect in one path on at least two vertices. Since $G' \in \mathcal{K}$, by Lemma 2.12, G' has a subdivided chordal path, so (A) is true. Statement (B) then follows from Theorem 2.10 and the induction hypothesis.

Since the parameters M and Z sum over connected components, the result for every $G \in \mathcal{K}$ that does not have a cut-edge follows from the result for connected graphs. □

Since $\mathcal{K}$ includes all complete subdivision graphs of simple graphs and multigraphs, we have the following corollary.

Corollary 2.17. *If G is a simple graph or multigraph that does not have a cut-edge, then* $\mathrm{M}(F,\mathring{\overline{G}}) = \mathrm{Z}(\mathring{\overline{G}})$.

3. Zero forcing number and maximum nullity of edge subdivision graphs. Recall that in [3], the authors ask the following question: Suppose G is any graph in which each vertex has degree at least 3 and H is a graph that has one less edge subdivision than $\mathring{\overline{G}}$. Is it always the case that $M(H) < M(\mathring{\overline{G}})$? The graphs G and H given in Example 3.1 below provide a negative answer to this question. We use the following well known observation: If $G = \cup_{i=1}^{h} G_i$, $G_i = (V_i, E_i)$, and (F is infinite or $E_i \cap E_j = \emptyset$ for $i \neq j$), then $\mathrm{mr}(F,G) \leq \sum_{i=1}^{h} \mathrm{mr}(F,G_i)$.

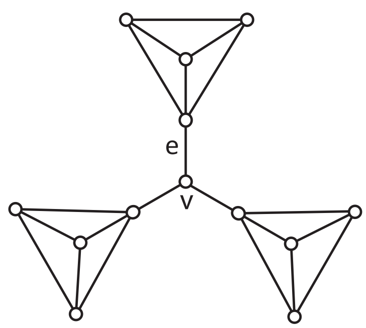


Fig. 3.1: A graph G that provides negative answer to Question 1.7.

Example 3.1. Let G be the graph in Figure 3.1, which is the connected union of three copies of K_4 (the complete graph on four vertices) and the star graph $K_{1,3}$, with these graphs having no common edges and the copies of K_4 disjoint; the edge e is one of the edges of the $K_{1,3}$. Let H be the graph that has one less edge subdivision than $\mathring{\overline{G}}$ where the edge e in G is the only unsubdivided edge. The graphs $\mathring{\overline{G}}$ and H are shown in Figure 3.2.

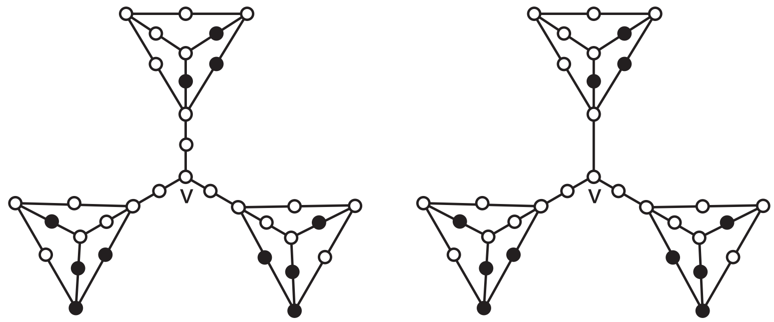
Since K_4 has a Hamiltonian path, by Theorem 1.9, $\mathrm{mr}(F,\mathring{\overline{K_4}}) = 6$. The subgraph $K_{1,3}$ is a tree. Hence, by Theorem 1.4, $\mathrm{M}(F,\mathring{\overline{K_{1,3}}}) = \mathrm{P}(\mathring{\overline{K_{1,3}}}) = 2$, so $\mathrm{mr}(F,\mathring{\overline{K_{1,3}}}) = 5$. Let L be the graph obtained from $K_{1,3}$ by subdividing all but one edge; again by Theorem 1.4, $\mathrm{M}(L) = \mathrm{P}(L) = 2$ and so $\mathrm{mr}(F,L) = 4$. Since $\mathring{\overline{G}}$ is a union of three copies of $\mathring{\overline{K}}_4$ and one copy of $\mathring{\overline{K}}_{1,3}$,

$$\mathrm{mr}(F,\mathring{\overline{G}}) \leq 3\,\mathrm{mr}(F,\mathring{\overline{K_4}}) + \mathrm{mr}(F,\mathring{\overline{K_{1,3}}}) = 23 \text{ and } \mathrm{M}(F,\mathring{\overline{G}}) \geq 34 - 23 = 11.$$

Similarly, H is a union of three copies of $\mathring{\overline{K}}_4$ and one copy of L so

$$\mathrm{mr}(F,H) \leq 3\,\mathrm{mr}(F,\mathring{\overline{K_4}}) + \mathrm{mr}(F,Ł) = 22 \text{ and } \mathrm{M}(F,H) \geq 33 - 22 = 11.$$

Furthermore, zero forcing sets of order 11 for both $\mathring{\overline{G}}$ and H are exhibited in Figure 3.2. Therefore, $\mathrm{M}(F,H) = \mathrm{Z}(H) = \mathrm{M}(F,\mathring{\overline{G}}) = \mathrm{Z}(\mathring{\overline{G}}) = 11$.

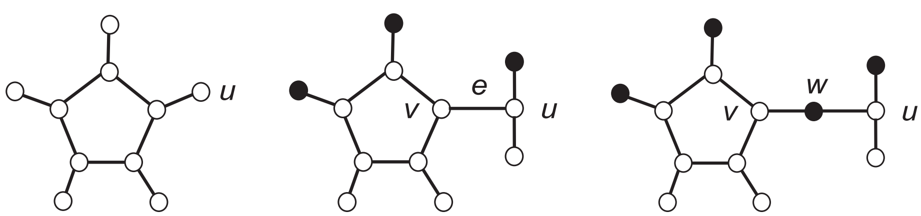


(a) Complete subdivision graph of G (b) H

Fig. 3.2: The complete subdivision graph of G and the graph H.

Given that we conjecture $\mathrm{M}(F,\hat{\overrightarrow{G}}) = \mathrm{Z}(\hat{\overrightarrow{G}})$ for every field F and graph G, one might be tempted to think that subdividing an edge cannot increase the difference $\mathrm{Z}(G) - \mathrm{M}(F,G)$. The next example shows that this is not the case. In fact, $\mathrm{M}(F,G) = \mathrm{Z}(G)$ does not necessarily imply $\mathrm{M}(F,G_e) = \mathrm{Z}(G_e)$.

Example 3.2. The pentasun H_5 is a five cycle with a degree one neighbor attached to each cycle vertex, shown in Figure 3.3(a). The graph G in Figure 3.3(b) is obtained from H_5 by adding two degree one neighbors of u, where u is a vertex of degree one in H_5. Note the labeled edge $e = uv$; the result G_e of subdividing edge e is shown in Figure 3.3(c). We show that $\mathrm{M}(F,G) = \mathrm{Z}(G)$ but $\mathrm{M}(F,G_e) < \mathrm{Z}(G_e)$.



(a) H_5 (b) G (c) $G' = G_e$

Fig. 3.3: The graphs for Example 3.2

It is well known that $\mathrm{M}(F,H_5) = 2$, $\mathrm{M}(F,H_5 - u) = 2$, $\mathrm{Z}(H_5) = 3$, and $\mathrm{Z}(H_5 - u) = 2$. Let $G' := G_e$. The maximum nullity of G and G' can be obtained by performing cut-vertex reduction using vertex v. Let W_1 (respectively, W_1') be the vertices in the component of $G - v$ (respectively, G') containing u and let W_2 (respectively, W_2') be the vertices of the other component For $i = 1, 2$, let $G_i = G[W_i \cup \{v\}]$ and $G_i' = G[W_i' \cup \{v\}]$. So, $\mathrm{mr}(F, G_1) = 2$, $\mathrm{mr}(F, G[W_1]) = 2$, $\mathrm{mr}(F, G_2) = 7$, $\mathrm{mr}(F, G[W_2]) = 6$, $\mathrm{mr}(F, G_1') = 3$, $\mathrm{mr}(F, G'[W_1']) = 2$, $\mathrm{mr}(F, G_2') = 7$, and $\mathrm{mr}(F, G'[W_2']) = 6$. Thus,

$$\mathrm{mr}(F,G) = \sum_{i=1}^{2} \mathrm{mr}(F, G[W_i]) + \min\{2, \sum_{i=1}^{2} \mathrm{r}_v(F, G_i)\} = 9 \text{ so } \mathrm{M}(F,G) = 12 - 9 = 3$$

and

$$\mathrm{mr}(F,G') = \sum_{i=1}^{2} \mathrm{mr}(F,G'[W_i']) + \min\{2, \sum_{i=1}^{2} \mathrm{r}_v(F,G_i')\} = 10 \text{ so } \mathrm{M}(F,G_e) = \mathrm{M}(F,G') = 13-10 = 3.$$

Zero forcing sets of size 3 for G and 4 for G_e are exhibited in Figures 3.3(b) and 3.3(c), and it is not difficult to see that no smaller sets can force. Thus $\mathrm{M}(F,G) = \mathrm{Z}(G) = 3$ and $\mathrm{M}(F,G_e) = 3 < \mathrm{Z}(G_e) = 4$. Zero forcing number and maximum nullity can also be computed by the minimum rank software [4].

It is easy two see that there is no relationship between the change in maximum nullity and the change in zero forcing number of G and G_e. In Example 3.2 edge subdivision increased zero forcing number but not maximum nullity. Subdividing any cycle edge of the pentasun H_5 increases maximum nullity but not zero forcing number (this follows from Proposition 2.1).

4. Path cover number of edge subdivision graphs.

In this section we investigate the effects of edge subdivisions on the path cover number.

PROPOSITION 4.1. *Let G be a graph and e an edge of G. Then*

$$\mathrm{P}(G) \le \mathrm{P}(G_e) \le \mathrm{P}(G) + 1.$$

If there exists a minimum path cover $\mathcal{P}$ of G such that e is on a path in $\mathcal{P}$, then $\mathrm{P}(G_e) = \mathrm{P}(G)$.

Proof. Let $e = uv$ and let w be the new vertex in G_e that is adjacent to u and v. We first prove the upper bounds. Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a minimum path cover of G. If e is in a path $Q = P_i$ for some $i = 1 \dots k$, then $(\mathcal{P} \setminus \{Q\}) \cup \{Q_e\}$ is a path cover of G_e, and so $\mathrm{P}(G_e) \le \mathrm{P}(G)$. If e is not in any P_i, then $\mathcal{P} \cup \{w\}$ is a path cover of G_e. In either case, $\mathrm{P}(G_e) \le \mathrm{P}(G) + 1$.

To prove the lower bound on $\mathrm{P}(G_e)$, let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a minimum path cover of G_e. Then $w \in P_i$ for some i. If $\{w\} = P_i$, then $\mathcal{P} \setminus \{P_i\}$ is a path cover of G. If the edges uw and wv are in P_i, define P_i' to be the path obtained from P_i by removing uw and wv, and then adding the edge uv. Then $(\mathcal{P} \setminus \{P_i\}) \cup \{P_i'\}$ is a path cover of G. If w is an endpoint of $P_i \ne \{w\}$, define P_i' to be the path P_i with w removed. Then $(\mathcal{P} \setminus \{P_i\}) \cup \{P_i'\}$ is a path cover of G. In all cases, $\mathrm{P}(G) \le \mathrm{P}(G_e)$. □

PROPOSITION 4.2. *Let G be a graph and let e be an edge of G. If e is incident to a vertex of degree at most 2, then* $\mathrm{P}(G_e) = \mathrm{P}(G)$.

Proof. By Proposition 4.1, $\mathrm{P}(G) \le \mathrm{P}(G_e)$. Now it remains to show that $\mathrm{P}(G_e) \le \mathrm{P}(G)$. Let $e = uv$ and let w be the new vertex that is adjacent to u and v in G_e. Without loss of generality, let $\deg u \le 2$. Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a minimum path cover of G. If e is on some path P_i in $\mathcal{P}$, then by Proposition 4.1, $\mathrm{P}(G) = \mathrm{P}(G_e)$. If e is not in any P_i, then u is the endpoint of some path in $\mathcal{P}$. Without loss of generality, say u is in P_1, then let P_1' be the path obtained by adding w to P_1. Then $(\mathcal{P} \setminus \{P_1\}) \cup \{P_1'\}$ is a path cover of G_e. In either case, $\mathrm{P}(G_e) \le \mathrm{P}(G)$. □

It is conjectured that for all graphs G, $\mathrm{M}(F, \hat{\overline{G}}) = \mathrm{Z}(\hat{\overline{G}})$. The following is an example of a graph G with $\mathrm{P}(\hat{\overline{G}}) < \mathrm{Z}(\hat{\overline{G}})$.

EXAMPLE 4.3. Let G be the graph pictured in Figure 4.1, called a double triangle. Since G contains a Hamiltonian path, by Theorem 1.9, $\mathrm{Z}(\mathring{\overrightarrow{G}}) = \mathrm{M}(F, \mathring{\overrightarrow{G}}) = 3$. However, $P(\mathring{\overrightarrow{G}}) = 2$ because $\mathring{\overrightarrow{G}}$ is not a path and a path cover of order 2 is exhibited in Figure 4.1.

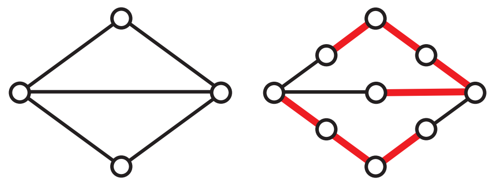


Fig. 4.1: A double triangle and its complete subdivision graph.

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