SPN GRAPHS AND RANK-1 CP-COMPLETABLE GRAPHS*

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Abstract. A simple graph G is an SPN graph if every copositive matrix having graph G is the sum of a positive semidefinite and nonnegative matrix. SPN graphs were introduced in [Shaked-Monderer, SPN graphs: When copositive = SPN, *Linear Algebra Appl.*, 509(15):82–113, 2016], where it was conjectured that the complete subdivision graph of K_4 is an SPN graph. We disprove this conjecture, which in conjunction with results in the Shaked-Monderer paper show that a subdivision of K_4 is a SPN graph if and only if at most one edge is subdivided. We conjecture that a graph is an SPN graph if and only if it does not have an F_5 minor, where F_5 is the fan on five vertices. To establish that the complete subdivision graph of K_4 is not an SPN graph, we introduce rank-1 CP completions and characterize graphs that are rank-1 CP completable.

Keywords: Copositive, SPN, rank-1 CP completion, F_5, fan, matrix, graph

AMS subject classification 15B48, 05C50, 15A83

1. Introduction.

A real symmetric matrix A is *copositive* if $\mathbf{x}^T A\mathbf{x} \geq 0$ for every nonnegative vector $\mathbf{x}$. A matrix A is *SPN* if it is a sum of a positive semidefinite matrix and a symmetric nonnegative one. Every SPN matrix is copositive but not conversely. Determining whether a real symmetric matrix is copositive is hard (in complexity terms this problem is co-NP complete), while determining whether a matrix is SPN is easier (it can be done by a semidefinite program). Patterns of nonzero entries can play a role. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, the *graph of* A is $G(A) = ([n], E)$ where $ij \in E$ if and only if $i \neq j$ and a_{ij} is non-zero ($[n]$ denotes the set $\{1, \dots, n\}$). It was suggested by Shaked-Monderer [4] to characterize all graphs G with the property that a symmetric matrix A with $G(A) = G$ is copositive if and only if it is SPN. Such a graph is called an *SPN graph*; a graph that does not have this property is called *non-SPN*. In [4] the study of SPN graphs was initiated and numerous results that specific graphs were or were not SPN were established (see also the corrigendum [5]). In this paper we conjecture a characterization of SPN graphs, identify what needs to be done to establish this characterization, and make progress towards the proof of the following conjecture.

Conjecture 1.1. *A graph G is an SPN graph if and only if G does not have the fan graph F_5 shown in Figure 1.1 as a minor.*

Discussion and additional related conjectures are in Section 5; definitions of graph theory terms are given in Section 2. A main result established is the following.

Theorem 1.2. *The complete subdivision $\mathring{\overline{K}}_4$ is a non-SPN graph.*

Theorem 1.2 shows that [4, Conjecture 9.13] is false. The next result follows from [4, Lemma 7.2] for SPN graphs, and [4, Lemma 7.2] and Theorem 1.2 for non-SPN graphs.

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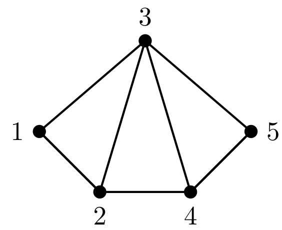

FIGURE 1.1. *The fan F_5*

COROLLARY 1.3. *A subdivision of K_4 is an SPN graph if and only if at most one of the original edges was subdivided.*

Theorem 1.2 is established in Section 4 by introducing the theory of rank-1 CP completions in Section 3, where we characterize graphs G that have the property that every partial rank-1 CP matrix A with specified entries corresponding to the edges of G can be completed to a rank-1 CP matrix.

2. Preliminaries. A (simple) *graph* G is a pair (V, E), where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges, i.e., two element subsets of V. The edge $\{i, j\}$ is often denoted by ij. A graph $H = (V(H), E(H))$ is a *subgraph* of $G = (V(G), E(G))$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a graph G and $e = uv \in E(G)$, subdividing edge e creates a new graph G_e from G by adding a new vertex w adjacent to u and v and deleting e, so $G_e = (V(G) \cup \{w\}, (E(G) \setminus \{e\}) \cup \{uw, vw\})$. Any graph obtained by successively subdividing edges of G is a *subdivision* of G. The graph obtained from G by subdividing each edge once is denoted by $\overset{-\circ-}{G}$. A vertex v of a graph G is a *cut vertex* of G if deleting v disconnects G. A *block* of G is a subgraph that has no cut vertex and is maximal with respect to this property. A graph is 2-connected if it has at least three vertices and does not have a cut vertex. The *contraction* of an edge $e = uv$ of G is obtained by identifying the vertices u and v, deleting any loops that arise in this process, and replacing any multiple edges by a single edge. A graph H is a *minor* of a graph G if H can be obtained from G by a series of the following operations (in any order): delete an isolated vertex; delete an edge; contract an edge.

A signed graph is a graph of the form $\mathcal{G} = (V, E, \Sigma)$ where V is the set of vertices, E is the set of edges, and $\Sigma : E \to \{+, -\}$ signs the edges. The *signed graph of a symmetric matrix* A, denoted by $\mathcal{G}(A)$, is obtained from $G(A)$ by assigning the sign of a_{ij} to the edge ij. A signed subgraph $\mathcal{H}$ of a signed graph $\mathcal{G}$ is a subgraph $\mathcal{H}$ of $\mathcal{G}$ with $\Sigma(\mathcal{H})$ being the restriction of $\Sigma(\mathcal{G})$ to $E(\mathcal{H})$. In signed graph drawings, a dashed line denotes a negative edge and a solid line a positive edge. For a graph $G = (V, E)$ (or a signed graph $\mathcal{G} = (V, E, \Sigma)$) and $S \subseteq V$, $G[S] = (S, E[S])$ (or $\mathcal{G}[S] = (S, E[S], \Sigma[S])$) is the *induced* subgraph of G (or $\mathcal{G}$) on the vertices S, where $E[S] = \{uv \in E : u, v \in S\}$ (and $\Sigma[S]$ is the restriction of Σ to $E[S]$). For a signed graph $\mathcal{G}$ and such S, $G_-[S]$ denotes the graph on vertices S whose edges are the negative edges of $\mathcal{G}[S]$.

The space of $n \times n$ real symmetric matrices is denoted by $\mathcal{S}_n$. The set of all copositive matrices in $\mathcal{S}_n$ is denoted by $\mathcal{COP}_n$. This is a closed convex cone in $\mathcal{S}_n$, and it contains the closed convex cone $\mathcal{SPN}_n$ consisting of the SPN matrices of order n. The inclusion $\mathcal{SPN}_n \subseteq \mathcal{COP}_n$ is an equality if and only if $n \leq 4$.

The cone $\mathcal{COP}_5$, as the first to differ from its subcone $\mathcal{SPN}_5$, has been studied in [1, 3]. As in [1], we define

$$S(\boldsymbol{\theta}) = \begin{pmatrix} 1 & -\cos(\theta_1) & \cos(\theta_1+\theta_2) & \cos(\theta_4+\theta_5) & -\cos(\theta_5) \\ -\cos(\theta_1) & 1 & -\cos(\theta_2) & \cos(\theta_2+\theta_3) & \cos(\theta_1+\theta_5) \\ \cos(\theta_1+\theta_2) & -\cos(\theta_2) & 1 & -\cos(\theta_3) & \cos(\theta_3+\theta_4) \\ \cos(\theta_4+\theta_5) & \cos(\theta_2+\theta_3) & -\cos(\theta_3) & 1 & -\cos(\theta_4) \\ -\cos(\theta_5) & \cos(\theta_1+\theta_5) & \cos(\theta_3+\theta_4) & -\cos(\theta_4) & 1 \end{pmatrix}$$

for $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$. For $\boldsymbol{\theta} > \mathbf{0}$ such that $\sum_{i=1}^{5} \theta_i < \pi$ the matrices $S(\boldsymbol{\theta})$ are (extremal) *Hildebrand matrices* [3], $S(\mathbf{0})$ is the (extremal) *Horn matrix*, and for other $\boldsymbol{\theta} \geq \mathbf{0}$ with $\sum_{i=1}^{5} \theta_i < \pi$ the matrix $S(\boldsymbol{\theta})$ is a copositive matrix that is not SPN [1].

The submatrix of the $n \times n$ matrix A in rows indexed by $R \subseteq [n]$ and columns indexed by $C \subseteq [n]$ is denoted by $A[R|C]$. When $R = C$, we write $A[R]$ for $A[R|R]$, and $A[R]$ is called a *principal submatrix* of A. We sometimes omit the set brackets of R and C in these notations, writing $A[2]$ instead of $A[\{2\}]$, for example. If $M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ and A is nonsingular, the *Schur complement* of A in M is $M/A = C - B^T A^{-1} B$; a permutation similarity is applied to define the Schur complement $A/A[S]$ for $S \subset [n]$.

We will use the following results from the literature.

LEMMA 2.1. [4, Lemma 3.3(b)] *Let $A = \begin{pmatrix} c & \mathbf{a}^T \\ \mathbf{a} & B \end{pmatrix}$ with $c > 0$ and $\mathbf{a} \leq 0$. Then $A \in \mathcal{SPN}_n$ (respectively, $A \in \mathcal{COP}_n$) if and only if $A/A[1] \in \mathcal{SPN}_{n-1}$ (respectively, $A/A[1] \in \mathcal{COP}_{n-1}$).*

LEMMA 2.2. [4, Lemma 8.1] *Let $\mathcal{G}$ is a signed graph, and let $\widehat{\mathcal{G}}$ be a signed graph obtained by subdividing a negative edge of $\mathcal{G}$, replacing it by a negative path of length 2. Then $\mathcal{G}$ is SPN if and only if $\widehat{\mathcal{G}}$ is SPN.*

3. Rank-1 CP completion.

In Section 4 we will use rank-1 CP matrix completions (Theorem 3.9) in the proof that the complete subdivision of K_4 is not SPN. In this section we define rank-1 CP-completable graphs, establish some of their properties, and characterize rank-1 CP-completable graphs, in addition to proving Theorem 3.9.

Recall that a real symmetric matrix A is *completely positive* (*CP*) if $A = BB^T$, where $B \geq 0$. A rank-1 completely positive matrix is of the form $\mathbf{b}\mathbf{b}^T$, where $\mathbf{0} \neq \mathbf{b} \geq \mathbf{0}$.

A *partial matrix* is a matrix in which some entries specified and others are not (no entries specified or all entries specified are permitted). We use $A^?$ to denote a partial matrix, and a_{ij} to denote a specified entry of A; ? can be used to denote an unspecified entry. An $n \times n$ matrix $B = [b_{ij}]$ is a *completion* of a partial matrix $A^?$ if $b_{ij} = a_{ij}$ for every specified entry a_{ij} of $A^?$. A partial matrix is symmetric if whenever a_{ij} is specified, so is a_{ji} and $a_{ji} = a_{ij}$. For a graph G, a *partial matrix described by* G is a symmetric partial matrix that has the i,j-entry specified if and only if ij is an edge of G. We note that this is a different association of a graph and (partial) matrices than that used throughout the rest of the paper, where matrices are complete and an nonedge of G is associated with a *zero* off-diagonal entry of the matrix.

A symmetric partial matrix is a *partial rank-*1 *CP matrix* if every specified entry is positive and all diagonal entries are unspecified. A graph G is *rank-*1 *CP-completable* if every partial rank-1 CP matrix described by G can be completed to a rank-1 CP matrix. A graph G is *uniquely rank-*1 *CP-completable* if every partial rank-1 CP matrix described by G has a unique rank-1 CP completion.

Note that our definition of rank-1 CP-completable is designed for the purpose of showing certain graphs are non-SPN and is different from the definition of CP-completable in [2], where the diagonal is fully specified and any fully specified principal submatrix is required to be CP in order for the partial matrix to be considered a partial CP matrix (in addition to being symmetric and having all specified off-diagonal entries nonnegative). The differences in the definitions arise naturally from the differences in problems being considered. Without the rank-1 restriction, every partial CP matrix with unspecified diagonal can be completed to a CP matrix, whereas we need the freedom to set the diagonal for our application in Theorem 3.9.

A graph G is the *vertex sum (at v)* of G_1 and G_2 if $G = G_1 \cup G_2$, $G_1 \cap G_2 = \{v\}$ is a single vertex, and $G \neq G_i, i = 1, 2$. If G is the vertex sum at v of G_1 and G_2, then v is a *cut vertex* of G. A cycle is *odd* or *even* according as its order is odd or even. We start with a few simple observations.

Remark 3.1. Clearly a graph of order one is rank-1 CP-completable (and not uniquely so). Any partial rank-1 CP matrix described by K_2 has the form $\begin{pmatrix} ? & a \\ a & ? \end{pmatrix}$, $a > 0$, and $A_x = \begin{pmatrix} x & a \\ a & \frac{a^2}{x} \end{pmatrix}$ is a rank-1 CP completion of $A^?$ for any $x > 0$. This form describes all the rank-1 CP completions of K_2, which is therefore rank-1 CP-completable but not uniquely so.

A disjoint union of rank-1 CP-completable graphs is also rank-1 CP-completable: If $\mathbf{a}\mathbf{a}^T$, $\mathbf{a} \geq \mathbf{0}$, is a rank-1 CP completion of $A^?$ and $\mathbf{b}\mathbf{b}^T$, $\mathbf{b} \geq \mathbf{0}$, is a is a rank-1 CP completion of $B^?$, then $\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}^T$ is a rank-1 CP completion of $A^? \oplus B^?$.

Finally, note that if $\mathbf{b}\mathbf{b}^T$ is a rank-1 CP completion of a partial rank-1 matrix described by a graph G, and $b_i = 0$, then the vertex i is necessarily an isolated vertex of G, and setting $b_i = x$ for any $x > 0$ will also yield a rank-1 CP completion of the same matrix. Thus if a graph G is rank-1 CP-completable, any partial rank-1 CP matrix described by G has a completion $\mathbf{b}\mathbf{b}^T$ with $\mathbf{b}$ a *positive* vector.

Proposition 3.2. *If H is a subgraph of a rank-1 CP-completable graph G, then H is rank-1 CP-completable.*

Proof. Let $A^?$ be a partial rank-1 CP matrix described by H. For every $ij \in E(G) \setminus E(H)$ choose any positive number for a_{ij}. This creates a a partial rank-1 CP matrix $B^?$ described by G. A rank-1 CP completion B of $B^?$ yields a rank-1 CP-completion of $A^?$ (the principal submatrix of B on H's vertices). ☐

Lemma 3.3. *Suppose that G is the vertex sum of K_2 and H. If H is rank-1 CP-completable (respectively, uniquely rank-1 CP-completable), then G is rank-1 CP-completable (uniquely rank-1 CP-completable).*

Proof. Without loss of generality assume $V(K_2) = \{1, 2\}$ and $V(H) = \{2, \ldots, n\}$.Then a partial rank-1 CP matrix described by G has the form:

$$A^? = \begin{pmatrix} ? & a & ? & \cdots & ? \\ a & & & & \\ ? & & & & \\ \vdots & & & B^? & \\ ? & & & & \end{pmatrix},$$

where $a > 0$ and $B^?$ is an partial rank-1 CP matrix described by H. Let $B = \mathbf{b}\mathbf{b}^T$ be a completion of $B^?$, where $\mathbf{b}^T = (b_2, \ldots, b_n)$ is a positive vector. Then appending $b_1 = \frac{a}{b_2}$ to $\mathbf{b}$ we get that

$$\begin{pmatrix} b_1 \\ \mathbf{b} \end{pmatrix} \begin{pmatrix} b_1 \\ \mathbf{b} \end{pmatrix}^T$$

is a rank-1 CP completion of $A^?$. If the completion B of $B^?$ is unique, then so is this completion. ∎

For any partial rank-1 CP matrix A such that A is described by a connected graph G, Algorithm 3.4 below provides a method to determine whether or not A has a rank-1 CP completion and finds all such completions. A *rooted tree* is a tree with a distinguished vertex, called the *root*. The *parent* of a vertex v (that is not the root) in a rooted tree is the neighbor of v in the path between v and the root.

ALGORITHM 3.4. *Given a partial rank-1 CP matrix $A^?$ such that $A^?$ is described by a connected graph G, find all possible rank-1 CP completions of $A^?$ or determine there are none.*

1. *Initialize:*
   (a) *Choose a spanning tree T of G and a root, k.*
   (b) *Denote every unspecified i,j-entry of A by x_{ij}.*
   (c) *Set the true/false variable COMPLETION :=* `true`.
2. *Assign a (positive) real variable x to the diagonal entry x_{kk} corresponding to the root: $x_{kk} := x$.*
3. *Proceeding outwards from the root, successively assign each diagonal entry x_{ii} from its parent x_{jj} and a_{ij}: $x_{ii} := \frac{a_{ij}^2}{x_{jj}}$.*
4. *Assign every unspecified off diagonal entry x_{ij} from the diagonal entries of the associated 2×2 submatrix: $x_{ij} := \sqrt{x_{ii}x_{jj}}$.*
5. *For every edge ij of G such that ij is not an edge of T, compare a_{ij} to x_{ij}:*
   (a) *Case: x is still a variable*
      i. *Case: $a_{ij} = x_{ij}$ can be solved for x*
         A. *Solve $a_{ij} = x_{ij}$ for x and assign the resulting value to x.*
         B. *Update all entries x_{st} accordingly.*
      ii. *Case: The equation $a_{ij} = x_{ij}$ is true for all values of x*
         No action taken.
      iii. *Case: The equation $a_{ij} = x_{ij}$ is false for all values of x*
         A. *Set COMPLETION :=* `false`.
         B. *Exit Step 5.*
   (b) *Case: x has already been assigned a numerical value*
      i. *Case: $a_{ij} = x_{ij}$*
         No action taken.
      ii. *Case: $a_{ij} \neq x_{ij}$*
         A. *Set COMPLETION :=* `false`.
         B. *Exit Step 5.*

PROPOSITION 3.5. *When Algorithm 3.4 is applied to a partial rank-1 CP matrix $A^?$ such that $A^?$ is described by a connected graph G, the outcome produced by Algorithm 3.4 is correct:*

- *If COMPLETION =* `true`*, then all possible completions have been found (either a one parameter family with positive real variable x or a unique rank-1 CP completion).*
- *If COMPLETION =* `false`*, then $A^?$ does not have a rank-1 CP completion.*

Furthermore, Step 5(a)i occurs at most once. When Step 5(a)i occurs, there is no longer any variable and the completion, if it exists, is unique.

Proof. It is clear that COMPLETION = `false` implies $A^?$ does not have a rank-1 CP completion. So assume COMPLETION = `true`. By Lemma 3.3, the completion produced in Steps 2 and 3 is a rank-1 CP completion of the entries of $A^?$ described by T, and A is a completion of $A^?$ by Step 5. ∎

Lemma 3.6. *A cycle C_n is rank-1 CP-completable if and only if n is odd; in this case it is uniquely rank-1 CP-completable.*

Proof. Label the vertices of C_n in cycle order as $1, 2, \ldots, n$ and let $A^?$ be a partial rank-1 CP matrix described by the cycle; the specified entries of $A^?$ are $a_{i,i+1}$ for $i = 1, \ldots, n-1$ and $a_{1,n}$. Apply Algorithm 3.4, choosing the path $1, 2, \ldots, n$ as the spanning tree and 1 as its root in Step 1.

In Step 2 we get that

$$x_{ii} = \begin{cases} \frac{a_{1,2}^2 a_{3,4}^2 \cdots a_{2k-1,2k}^2}{x a_{2,3}^2 a_{4,5}^2 \cdots a_{2k-2,2k-1}^2} & \text{if } i = 2k, \\ \frac{x a_{2,3}^2 a_{4,5}^2 \cdots a_{2k,2k+1}^2}{a_{1,2}^2 a_{3,4}^2 \cdots a_{2k-1,2k}^2} & \text{if } i = 2k+1. \end{cases}$$

for $i = 2, \ldots, n$.

Then in Step 3 we get, in particular, that

$$x_{1,n} = \begin{cases} \frac{x a_{2,3} a_{4,5} \cdots a_{n-1,n}}{a_{1,2} a_{3,4} \cdots a_{n-2,n-1}} & \text{if } n \text{ is odd}, \\ \frac{a_{1,2} a_{3,4} \cdots a_{n-1,n}}{a_{2,3} a_{4,5} \cdots a_{n-2,n-1}} & \text{if } n \text{ is even}. \end{cases}$$

In Step 5 there is only one extra edge to check, namely $\{1, n\}$. For n odd, we solve the resulting equation and obtain

$$x = \frac{a_{1,2} a_{3,4} \ldots a_{n,1}}{a_{2,3} a_{4,5} \ldots a_{n-1,n}},$$

yielding a unique rank-1 CP completion for any $A^?$. For n even, $A^?$ has a rank-1 CP completion if and only if $a_{1,n} = x_{1,n} = \frac{a_{1,2} a_{3,4} \cdots a_{n-1,n}}{a_{2,3} a_{4,5} \cdots a_{n-2,n-1}}$, or equivalently,

$$\prod_{i=1}^{n/2} a_{2i-1,2i} = \prod_{i=1}^{n/2} a_{2i,2i \dotplus 1}$$

where $\dotplus$ denotes addition modulo n. In this case, a one-parameter family of rank-1 CP completions is obtained. ∎

Corollary 3.7. *Every graph of order at most three is rank-1 CP-completable.*

Theorem 3.8. *A connected graph G is rank-1 CP-completable if and only if G is a tree or G is unicyclic with an odd cycle. A unicyclic graph with an odd cycle is uniquely rank-1 CP-completable.*

Proof. Observe that any tree (of order at least two) can be constructed by starting with a K_2 and performing a (possibly empty) sequence of vertex sums with additional copies of K_2. Any unicyclic graph can be constructed from its cycle by performing a (possibly empty) sequence of vertex sums with copies of K_2. So a tree or a unicyclic graph with odd cycle is rank-1 CP-completable by Lemmas 3.3 and 3.6, and in the unicyclic case uniquely so.

By Proposition 3.2 and Lemma 3.6, any graph that contains an even cycle is not rank-1 CP-completable. Suppose a connected graph G has no even cycles and has at least two odd cycles $C^{(1)}$ and $C^{(2)}$. Then $C^{(1)}$ and $C^{(2)}$ are either disjoint or intersect in a single vertex. Let P be the shortest path from any vertex in $C^{(1)}$ to any vertex in $C^{(2)}$ (if $C^{(1)}$ and $C^{(2)}$ intersect in a single vertex, P is the common vertex). Without loss of generality, assume the vertices of $C^{(1)}$ are $1,\ldots,t$, the vertices of P are $t,\ldots,s$, and the vertices of $C^{(2)}$ are $s,\ldots,r$. Choose any positive values for $a_{i,i+1}$ for $1 \le i \le s-1$ and for $a_{t,1}$ to obtain $A^?[\{1,\ldots,s\}]$. Compute the unique rank-1 CP completion B of $A^?[\{1,\ldots,s\}]$. Choose positive values for $a_{i,i+1}$ for $s \le i \le r-1$ and for $a_{r,s}$ so that the unique rank-1 CP completion of $A^?[\{s,\ldots,r\}]$ has a value for the diagonal entry associated with vertex s that is different from b_{ss}. Then $A^?[\{1,\ldots,r\}]$ does not have a rank-1 CP completion. Thus, $G[\{1,\ldots,r\}]$ is not rank-1 CP-completable. By Proposition 3.2, G is not rank-1 CP-completable. □

The next theorem will be used in Section 4. It shows a connection between the rank-1 CP-completable graphs discussed in this section and the study of SPN (signed) graphs. Theorem 3.9 generalizes [4, Lemma 8.4]. See Figure 3.1 for illustrations of the graph transformation described in this theorem, generalizing the Λ-paw transformation described in [4]. In this figure, $\mathcal{H}$ is a non-SPN signed F_5 and $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ are non-SPN signed graphs that may be obtained by the generalized graph transformation in Theorem 3.9.

THEOREM 3.9. *Let $\mathcal{H} = (V, E, \Sigma)$ be a non-SPN signed graph. Let $S \subseteq V$. Let $\mathcal{G}$ be a signed graph obtained by adding a vertex v adjacent to each of the vertices in S by a negative edge, (possibly) deleting some of the negative edges of $\mathcal{H}[S]$, (possibly) changing some negative edges between vertices in S to positive edges, and adding a positive edge between each pair of vertices in S that are not adjacent in $\mathcal{H}$. If $H_-[S]$ is rank-1 CP-completable, then $\mathcal{G}$ is non-SPN.*

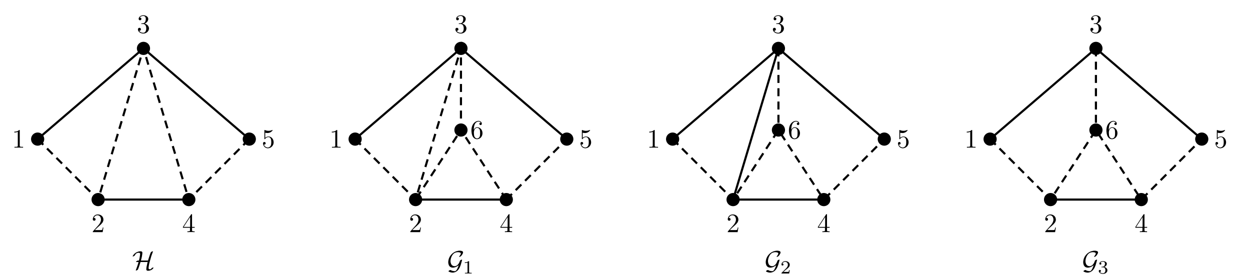

FIGURE 3.1. *Examples for Theorem 3.9 with $S = \{2, 3, 4\}$*

Proof. Observe that $G_-[S] \subseteq H_-[S]$ and S is a clique in $H \cup G$ (where H and G are the underlying graphs of $\mathcal{H}$ and $\mathcal{G}$). Without loss of generality assume $S = \{1,\ldots,k\}$, and $V(\mathcal{H}) = [n]$. Let A be a copositive non-SPN matrix with $\mathcal{G}(A) = \mathcal{H}$, and let $B^?$ be a partial rank-1 CP $k \times k$ matrix defined as follows on $H_-[S]$: For an edge ij of $H_-[S]$ that is not an edge of $\mathcal{G}$, set $b_{ij} = -a_{ij}$. For an edge ij of $H_-[S]$ that is a negative edge of $\mathcal{G}$, choose b_{ij} such that $0 < b_{ij} < -a_{ij}$. For an edge ij of $H_-[S]$ that is a positive edge of $\mathcal{G}$, choose $b_{ij} > -a_{ij}$. Then $B^?$ has a rank-1 CP completion $\mathbf{b}\mathbf{b}^T$ where $\mathbf{b} \in \mathbb{R}^k_+$ is a positive vector. Let $\mathbf{a} \in \mathbb{R}^n$ be the non-positive vector obtained by appending $n-k$ zeros to $-\mathbf{b}$. Consider the matrix

$$C = \begin{pmatrix} A + \mathbf{a}\mathbf{a}^T & \mathbf{a} \\ \mathbf{a}^T & 1 \end{pmatrix}.$$

Then, $C/C[n+1] = A$ is a non-SPN copositive matrix. Therefore, C is a non-SPN copositive matrix by Lemma 2.1, and $\mathcal{G}(C) = \mathcal{G}$. □

4. Subdivisions of K_4. In this section we show that the complete subdivision of K_4 is a non-SPN graph. Together with the results in [4], this shows that a subdivision of K_4 is an SPN graph if and only if at most one of the original edges was subdivided (that one edge may be subdivided as much as desired). We begin with some tools.

LEMMA 4.1. *For $0 < \alpha, \beta, \gamma < \pi/2$, the partial positive semidefinite matrix*

$$A^? = \begin{pmatrix} 1 & -\cos(\alpha) & \cos(\alpha+\beta) & ? \\ -\cos(\alpha) & 1 & -\cos(\beta) & \cos(\beta+\gamma) \\ \cos(\alpha+\beta) & -\cos(\beta) & 1 & -\cos(\gamma) \\ ? & \cos(\beta+\gamma) & -\cos(\gamma) & 1 \end{pmatrix}$$

has a unique positive semidefinite completion obtained by setting $? = -\cos(\alpha+\beta+\gamma)$.

Proof. Since

$$\begin{pmatrix} 1 & -\cos(\alpha) & \cos(\alpha+\beta) & -\cos(\alpha+\beta+\gamma) \\ -\cos(\alpha) & 1 & -\cos(\beta) & \cos(\beta+\gamma) \\ \cos(\alpha+\beta) & -\cos(\beta) & 1 & -\cos(\gamma) \\ -\cos(\alpha+\beta+\gamma) & \cos(\beta+\gamma) & -\cos(\gamma) & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 \\ -\cos(\alpha) \\ \cos(\alpha+\beta) \\ -\cos(\alpha+\beta+\gamma) \end{pmatrix} \begin{pmatrix} 1 \\ -\cos(\alpha) \\ \cos(\alpha+\beta) \\ -\cos(\alpha+\beta+\gamma) \end{pmatrix}^T + \begin{pmatrix} 0 \\ -\sin(\alpha) \\ \sin(\alpha+\beta) \\ -\sin(\alpha+\beta+\gamma) \end{pmatrix} \begin{pmatrix} 0 \\ -\sin(\alpha) \\ \sin(\alpha+\beta) \\ -\sin(\alpha+\beta+\gamma) \end{pmatrix}^T,$$

$A^?$ has a (rank-2) positive semidefinite completion. This is the unique positive semidefinite completion of $A^?$ because $A[\{1,2,3\}|\{4\}]$ must be orthogonal to $\ker A[\{1,2,3\}] = \operatorname{span}((\sin\beta, \sin(\alpha+\beta), \sin\alpha)^T)$ by the Column Inclusion Property of positive semidefinite matrices. ∎

PROPOSITION 4.2. *The signed graph $\mathcal{H}_5$ shown in Figure 4.1 is non-SPN. Specifically, with $\theta_i > 0$ for $i = 1, 2, 3, 4$ and $\sum_{i=1}^4 \theta_i < \frac{\pi}{2}$, the matrix*

$$(1) \qquad A = \begin{pmatrix} 1 & -\cos(\theta_1) & \cos(\theta_1+\theta_2) & 0 & 0 \\ -\cos(\theta_1) & 1 & -\cos(\theta_2) & \cos(\theta_2+\theta_3) & \cos(\pi/2+\theta_1) \\ \cos(\theta_1+\theta_2) & -\cos(\theta_2) & 1 & -\cos(\theta_3) & \cos(\theta_3+\theta_4) \\ 0 & \cos(\theta_2+\theta_3) & -\cos(\theta_3) & 1 & -\cos(\theta_4) \\ 0 & \cos(\pi/2+\theta_1) & \cos(\theta_3+\theta_4) & -\cos(\theta_4) & 1 \end{pmatrix},$$

is a non-SPN copositive matrix such that $\mathcal{G}(A) = \mathcal{H}_5$.

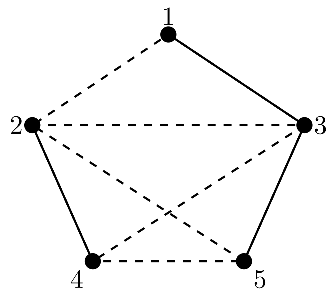

FIGURE 4.1. *The non-SPN signed graph* $\mathcal{H}_5$

Proof. Observe that $A \geq S(\boldsymbol{\theta})$, where $\boldsymbol{\theta}^T = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, with $\theta_5 = \frac{\pi}{2}$ (A differs from $S(\boldsymbol{\theta})$ only in the $1,4$ and $4,1$ entries). Since the Hildebrand matrix $S(\boldsymbol{\theta})$ is copositive, A is copositive.

Suppose that the matrix A is SPN. Then, $A \geq P$ for some positive semidefinite matrix P. Each of the submatrices $A[i, i \dotplus 1, i \dotplus 2]$ is positive semidefinite of rank 2 for $i = 1, 2, 3, 5$. The vector $\mathbf{v}^{(i)} = (\sin\theta_{i\dotplus 1}, \sin(\theta_i + \theta_{i\dotplus 1}), \sin\theta_i)^T$ spans the nullspace of $A[i, i \dotplus 1, i \dotplus 2]$ (with $\theta_5 = \frac{\pi}{2}$). Since $0 = {\mathbf{v}^{(i)}}^T A[i, i\dotplus 1, i\dotplus 2]\mathbf{v}^{(i)} \geq {\mathbf{v}^{(i)}}^T P[i, i\dotplus 1, i\dotplus 2]\mathbf{v}^{(i)} \geq 0$, and $\mathbf{v}^{(i)}$ is a positive vector, we get that $P[i, i\dotplus 1, i\dotplus 2] = A[i, i \dotplus 1, i \dotplus 2]$, $i = 1, 2, 3, 5$. Thus

$$P = \begin{pmatrix} 1 & -\cos(\theta_1) & \cos(\theta_1+\theta_2) & p & 0 \\ -\cos(\theta_1) & 1 & -\cos(\theta_2) & \cos(\theta_2+\theta_3) & \cos(\pi/2+\theta_1) \\ \cos(\theta_1+\theta_2) & -\cos(\theta_2) & 1 & -\cos(\theta_3) & \cos(\theta_3+\theta_4) \\ p & \cos(\theta_2+\theta_3) & -\cos(\theta_3) & 1 & -\cos(\theta_4) \\ 0 & \cos(\pi/2+\theta_1) & \cos(\theta_3+\theta_4) & -\cos(\theta_4) & 1 \end{pmatrix},$$

for some $-1 \leq p \leq 1$. Since P is positive semidefinite, so is $P[\{2, 3, 4, 5\}]$. Therefore, $\cos(\pi/2 + \theta_1) = -\cos(\theta_2 + \theta_3 + \theta_4)$ by Lemma 4.1, contradicting $\sum_{i=1}^4 \theta_i < \pi/2$ and $\theta_i > 0$ for $i = 1, 2, 3, 4$. Thus, no such positive semidefinite P exists and A is not SPN. □

We now prove Theorem 1.2.

Proof. Let $\mathcal{G}_i, i = 1, \ldots, 4$ be the graphs in Figure 4.2. Observe that $\mathcal{G}_4 = \overleftrightarrow{K_4}$. The graph $\mathcal{G}_1$ is $\mathcal{H}_5$, which is non-SPN by Proposition 4.2. We next show that $\mathcal{G}_2$ is non-SPN. Let A be a non-SPN matrix with graph $\mathcal{G}_1$, defined as in (1), where $\theta_i > 0$ for every $1 \leq i \leq 4$, $\sum_{i=1}^4 \theta_i < \frac{\pi}{2}$, and

$$\sin(\theta_1) < \frac{\cos(\theta_2)\cos(\theta_4)}{\cos(\theta_3)}.$$

Let

$$\mathbf{a}^T = (0, -\frac{\sin(\theta_1)}{\cos(\theta_4)}, -\cos(\theta_3), -1, -\cos(\theta_4)).$$

Then

$$B = \begin{pmatrix} A + \mathbf{a}\mathbf{a}^T & \mathbf{a} \\ \mathbf{a}^T & 1 \end{pmatrix}$$

is a copositive matrix (as a sum of the copositive $A \oplus 0$ and a rank-1 positive semidefinite matrix). Since the off-diagonal entries in the 6th column of A are nonpositive, $A = B/B[6]$ is also copositive by Lemma 2.1. Since A is non-SPN, B is non-SPN. As $\mathcal{G}(B) = \mathcal{G}_2$, $\mathcal{G}_2$ is non-SPN.

Since $\mathcal{G}_2$ is non-SPN, Theorem 3.9 (applied to $\mathcal{H} = \mathcal{G}_2$ and $S = \{2, 3, 6\}$) implies that the signed graph $\mathcal{G}_3$ is also non-SPN. The signed graph $\mathcal{G}_4$ is obtained from the non-SPN $\mathcal{G}_3$ by subdividing negative edges, and therefore is also non-SPN by Lemma 2.2. □

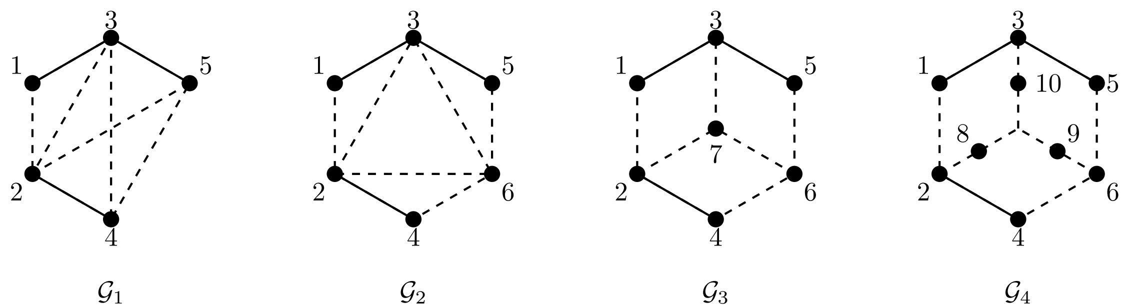

FIGURE 4.2. *Signed graphs used in the proof of Theorem 1.2*

In view of Corollary 1.3 and the discussion in [4, Remark 9.11], for a full description of SPN graphs (either by forbidden subgraphs or by their possible blocks) it remains to determine whether, or which, subdivisions of T_n are SPN, where T_n is the graph on n vertices consisting of $n-2$ triangles sharing a common base. So far it is only known that T_n is SPN for $n = 3, 4, 5$ and subdivisions of T_n are SPN for $n = 3, 4$ (see [4, 5]). This is discussed further in Section 5.

5. Conjectures. Here we list some conjectures, in increasing order of strength, leading up to Conjecture 1.1, which is the strongest. We also discuss possible methods to establish the conjectures.

CONJECTURE 5.1. *Any subdivision of a non-SPN graph is non-SPN.*

Conjecture 5.1 has been established in the case that the subdivided edge corresponds to a negative entry in a realizing non-SPN matrix in [4, Lemma 8.1] (see Lemma 2.2). So it remains to consider the case that the corresponding entry is positive.

If ζ is a graph property, and G has property ζ, then we say G is a ζ graph (e.g., SPN graph). A graph property ζ is *minor-closed* if G is a ζ graph implies that H is a ζ graph for every minor H of G. If ζ is minor-closed, then ζ-graphs are characterized by a finite set of forbidden graphs.

CONJECTURE 5.2. *Being an SPN graph is minor-closed, i.e., if H is a minor of an SPN graph G, then H is SPN.*

Since subgraphs of SPN graphs are SPN [4, Lemma 4.2], proving Conjecture 5.2 amounts to showing that the class of SPN graphs is closed under edge contractions. If Conjecture 5.2 is true, then the SPN graphs are characterized by a finite set of forbidden minors. Our Conjecture 1.1 is that the only forbidden minor is F_5.

From [4, Theorem 9.10] and Corollary 1.3 we obtain the next result.

Theorem 5.3. *Let G be an SPN graph. Then G does not contain the following subgraphs:*

(1) a subdivision of the fan F_5,
(2) a subdivision of CD_6, where CD_6 is the graph shown in figure 5.1.
(3) a subdivision of K_4 in which at least two edges were subdivided at least once each.

Since CD_6 and K_4 with two edges subdivided have an F_5 minor, every graph that is known to be non-SPN has an F_5 minor.

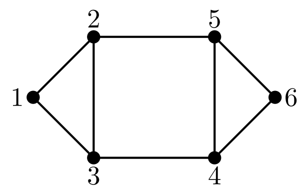

Figure 5.1. CD_6

Conjecture 1.1 is established for graphs on five vertices in [4, Theorem 7.3]; see also the corrigendum [5]. As discussed in [4, Remark 9.11], if a graph G is a 2-connected graph that contains no subdivision of F_5, no subdivision of CD_6, and no subdivision of K_4 where at least two edges have been subdivided, then G is either a subdivision of T_k, $k \geq 3$, or a DR_k, $k \geq 4$ (DR_k is a K_4 with one edge subdivided $k-4$ times). So if every subdivision of T_k is SPN, Conjecture 1.1 is true and SPN-graphs are determined. In this case, we would have not only the forbidden minor characterization in Conjecture 1.1, but also the block characterization suggested in [4], that a graph is SPN if and only if every block is one of an edge, a subdivision of T_k, $k \geq 3$, or a DR_k, $k \geq 4$.

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