# Computation of Minimal Rank and Path Cover Number for Certain Graphs

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**Abstract**

For a given undirected graph $G$, the minimum rank of $G$ is defined to be the smallest possible rank over all real symmetric matrices $A$ whose $(i, j)$th entry is nonzero whenever $i \neq j$ and $\{i, j\}$ is an edge in $G$. The path cover number of $G$ is the minimum number of vertex-disjoint paths occurring as induced subgraphs of $G$ that cover all the vertices of $G$. For trees, the relationship between minimum rank and path cover number is completely understood. However, for non-trees only sporadic results are known. We derive formulae for the minimum rank and path cover number for graphs obtained from edge-sums, and formulae for minimum rank of vertex sums of graphs. In addition we examine previously identified special types of vertices and attempt to unify the theory behind them.

## 1 Introduction

Spectral graph theory is the study of the eigenvalues of various matrices associated with graphs. In recent years there has been a great deal of interest in a *Symmetric Inverse Eigenvalue Problem*, concerning the study of possible eigenvalues of a real symmetric matrix whose nonzero entries are described by a given undirected graph.

All matrices discussed in this paper are real and symmetric. The graph $G(A)$ of an $n \times n$ matrix $A$ has $\{1, ..., n\}$ as vertices, and as edges the unordered pairs $\{i, j\}$ such that $a_{ij} \neq 0$ with $i \neq j$. Graphs $G$ of the form $G = G(A)$ do not have loops or multiple edges, and the diagonal of $A$ is ignored in the determination

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of $G(A)$. The aforementioned symmetric inverse eigenvalue problem asks: given a graph $G$, what eigenvalues are possible for a real symmetric matrix $A$ with $G(A) = G$? This is a very difficult problem, and complete solutions have been obtained only in special cases. Most of the work has focused on trees; our interest lies in extending some results to graphs in general.

For the matrix $A$, $\sigma(A)$ denotes the spectrum of $A$ and for $\lambda \in \sigma(A)$, $\mathrm{mult}_A(\lambda)$ denotes the multiplicity of $\lambda$. Define the following parameters of a graph G: $\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : G(A) = G\}$; $M(G) = \max\{\mathrm{mult}_A(\lambda) : \lambda \in \sigma(A) \text{ and } G(A) = G\}$; $P(G)$ is the *path cover number*, namely, the minimum number of vertex disjoint paths, occurring as induced subgraphs of $G$, that cover all the vertices of $G$; $\Delta(G)$ is the maximum of $p - q$ such that the deletion of $q$ vertices from $G$ leaves $p$ paths.

If we denote the order of $G$ by $|G|$, then it is easy to see that $|G| = M(G) + \mathrm{mr}(G)$, as noted in [BL]. This relation has been exploited to obtain results about the maximum possible multiplicity from results on the minimum rank, and also played a role in Johnson and Leal Duarte's result that, for trees, the three parameters $M(T)$, $P(T)$ and $\Delta(T)$ are equal [JLD99]. It follows from the proof given in [JLD99] that $\Delta(G) \leqslant M(G)$ for any graph. In this paper we show that for arbitrary graphs, $\Delta(G) \leqslant P(G)$ and give examples showing both $M(G) < P(G)$ and $P(G) < M(G)$ are possible (see Section 3). These results are obtained through a result allowing computation of the minimum rank of an edge-sum from the minimum rank of each of the pieces (see Section 2).

Let $G = (V, E)$ be a graph and let $v \in V$, $e \in E$. We denote by $G - e$ the subgraph of $G$ obtained by deleting edge $e$. We denote by $G - v$ the subgraph of $G$ obtained by deleting $v$ and all edges incident with $v$. Any induced subgraph of $G$ is obtained by deleting some subset of vertices. For a matrix $A$ with $G(A) = G$, the matrix $A(v)$ will denote the principal submatrix of $A$ obtained by deleting row and column $v$. In particular $G(A(v)) = G - v$. For the sake of completeness, in Section 5 we discuss the behavior of the parameters $M$, $P$, and $\Delta$ under induced subgraphs.

In this paper we make use of the following results. It is well-known that $\mathrm{mr}(G) = 1$ if and only if $G$ is $K_n$, the complete graph on $n$ vertices. Fiedler [F] established that $\mathrm{mr}(G) = n - 1$ if and only if $G$ is $P_n$, the path on $n$ vertices. Barrett and Loewy [BL] established that $\mathrm{mr}(G) = 2$ if and only if $G$ is not $K_n$, and does not contain as an induced subgraph any of the four "forbidden subgraphs": $P_4$, the complete tripartite graph $K_{3,3,3}$, and the two graphs shown in Figure 1 on the following page.

## 2 Vertex-sums of Graphs

We start by introducing a notion which will play a central role in all the following discussion.

**Definition 2.1** *Let $v$ a vertex of a graph $G$. The* rank-spread *of $G$ at $v$ is defined as $r_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G - v)$.*

Figure 1: Two forbidden subgraphs

We then have $0 \leqslant r_v(G) \leqslant 2$ (see, for example, [N]). In the following lemma we are interested in the matrices satisfying all of the following conditions:

$$A = \begin{bmatrix} a & \mathbf{b}^T \\ \mathbf{b} & A' \end{bmatrix}; \qquad G(A) = G; \qquad \mathbf{b} \in R(A'), \tag{1}$$

where $R(\cdot)$ denotes the range of a matrix.

**Lemma 2.2** *Let $G$ be a graph, and $v$ a vertex in $G$. If we assume $v = 1$, then*

*i.* $r_v(G) = 0$ *if and only if* $\min\{\operatorname{rank} A' \colon A$ *satisfies* $(1)\} = \operatorname{mr}(G - v)$*;*

*ii.* $r_v(G) = 1$ *if and only if* $\min\{\operatorname{rank} A' \colon A$ *satisfies* $(1)\} = \operatorname{mr}(G - v) + 1$*;*

*iii.* $r_v(G) = 2$ *otherwise.*

**Proof**

i. Let $A$ satisfy (1) with $\operatorname{rank} A' = \operatorname{mr}(G - v)$. Then $\tilde{A} = \begin{bmatrix} \mathbf{b}^T A'^{\dagger} \mathbf{b} & \mathbf{b}^T \\ \mathbf{b} & A' \end{bmatrix}$ satisfies (1) as well ($A^{\dagger}$ denotes the Moore-Penrose pseudoinverse). Now $\operatorname{mr}(G) \leqslant \operatorname{rank} \tilde{A} = \operatorname{rank} A' = \operatorname{mr}(G - v)$, so that $r_v(G) = 0$. Conversely, if $r_v(G) = 0$, any matrix $A$ with graph $G$ and rank equal to $\operatorname{mr}(G)$ will satisfy (1) with $\operatorname{rank} A' = \operatorname{mr}(G - v)$.

ii. Let $A$ satisfy (1) with $\operatorname{rank} A' = \operatorname{mr}(G-v)+1$. With regard to the matrix $\tilde{A}$ defined in (i.), we now have $\operatorname{mr}(G) \leqslant \operatorname{rank} \tilde{A} = \operatorname{rank} A' = \operatorname{mr}(G - v) + 1$, that is, $r_v(G) \leqslant 1$. Hence $r_v(G) = 1$, since 0 is excluded by (i.). Conversely, if $r_v(G) = 1$, by [N, Prop. 2.2] any matrix with graph $G$ and rank equal to $\operatorname{mr}(G)$ will satisfy (1) with $\operatorname{rank} A' = \operatorname{rank} A = \operatorname{mr}(G-v)+1$.

iii. Since $r_v(G) \leqslant 2$, the claim follows from (i.) and (ii.).

□

Let $G_1, \ldots, G_h$ be disjoint graphs. For each $i$, we select a vertex $v_i \in V(G_i)$ and join all $G_i$'s by identifying all $v_i$'s as a unique vertex $v$. The resulting graph is called the *vertex-sum* at $v$ of the graphs $G_1, \ldots, G_h$.

**Theorem 2.3** *Let $G$ be vertex-sum at $v$ of graphs $G_1, \dots, G_h$. Then*

$$r_v(G) = \min\left\{\sum_{i=1}^{h} r_v(G_i),\ 2\right\}, \tag{2}$$

*that is,* $\mathrm{mr}(G) = \sum_1^h \mathrm{mr}(G_i - v) + \min\left\{\sum_1^h r_v(G_i),\ 2\right\}$.

**Proof** By assuming $v = 1$, a matrix with graph $G$ can be written in the form

$$A = \begin{bmatrix} a & \mathbf{b}^T \\ \mathbf{b} & A' \end{bmatrix} = \begin{bmatrix} a & \mathbf{b}_1^T & \cdots & \mathbf{b}_h^T \\ \mathbf{b}_1 & A_1' & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_h & \mathbf{O} & \cdots & A_h' \end{bmatrix}, \tag{3}$$

where $G(A_i') = G_i - v$, $i = 1, \dots, h$. We will prove that $r_v(G) = 0$ if and only if $\sum_1^h r_v(G_i) = 0$, and that $r_v(G) = 1$ if and only if $\sum_1^h r_v(G_i) = 1$. Otherwise, since $r_v(G) \leqslant 2$, (2) follows.

Case I: let $r_v(G) = 0$. By Lemma 2.2, there exists a matrix $A$ of the form (3) such that $\mathbf{b} \in R(A')$, and $\operatorname{rank} A' = \mathrm{mr}(G - v) = \sum_1^h \mathrm{mr}(G_i - v)$. Therefore, for each $i$, $\mathbf{b}_i \in R(A_i')$ and $\operatorname{rank} A_i' = \mathrm{mr}(G_i - v)$. Thus, applying Lemma 2.2, we have $r_v(G_i) = 0$ for each $i$, hence, $\sum_1^h r_v(G_i) = 0$. Conversely, if $r_v(G_i) = 0$ for each $i$, we can find matrices $A_i = \left[\begin{smallmatrix} a_i & \mathbf{b}_i^T \\ \mathbf{b}_i & A_i' \end{smallmatrix}\right]$ satisfying (1) and $\operatorname{rank} A_i' = \mathrm{mr}(G_i - v)$. We can then derive a matrix $A$ as in (3), where $a$ can be any real number. Clearly $\mathbf{b} \in R(A')$ and $\operatorname{rank} A' = \mathrm{mr}(G - v)$. Therefore, again by Lemma 2.2, we conclude $r_v(G) = 0$.

Case II: let $r_v(G) = 1$. By case I, we then have $\sum_1^h r_v(G_i) \geqslant 1$. We now prove $\sum_1^h r_v(G_i) \leqslant 1$. Using Lemma 2.2, we can derive a matrix $A$ in the form (3) with $\mathbf{b} \in R(A')$ and $\operatorname{rank} A' = \sum_1^h \mathrm{mr}(G_i - v) + 1$. Therefore, there exists $j \in \{1, \dots, h\}$ such that $\operatorname{rank} A_j' = \mathrm{mr}(G_j - v) + 1$ and $\operatorname{rank} A_i' = \mathrm{mr}(G_i - v)$ for $i \neq j$. Thus, $\sum_1^h r_v(G_i) \leqslant 1$. Conversely, if $\sum_1^h r_v(G_i) = 1$, it suffices to modify slightly the proof of case I to obtain $r_v(G) = 1$. □

The next result is just a recasting of a special case of Theorem 2.3, which we state for completeness.

**Corollary 2.4** *Let $G$ be vertex-sum at $v$ of graphs $G_1$ and $G_2$. Then*

$$\mathrm{mr}(G_1) + \mathrm{mr}(G_2) - 2 \leqslant \mathrm{mr}(G) \leqslant \mathrm{mr}(G_1) + \mathrm{mr}(G_2),$$

*and both extremes are attainable.*

For attainment, join two stars at their centers (left-inequality) or join two paths at one of their ends (right-inequality).

By virtue of Theorem 2.3, we can determine the effect on the minimal rank by appending leaves (i.e., vertices of degree one) to a given graph.

**Lemma 2.5** *Let $G_1$ be a graph, and consider the graph $G$ obtained by appending $l$ leaves on a vertex $v$ of $G_1$. Then*

*i. if $l = 1$ and $r_v(G_1) = 0$, then $r_v(G) = 1$ and $\mathrm{mr}(G) = \mathrm{mr}(G_1) + 1$;*

*ii. otherwise, $r_v(G) = 2$ and $\mathrm{mr}(G) = \mathrm{mr}(G_1) + 2 - r_v(G_1)$.*

**Proof** Let us denote the leaves by the graphs $G_2, \dots, G_{l+1}$. Note that, for each $i = 2, \dots, l+1$, $r_v(G_i) = 1$, while $\mathrm{mr}(G_i - v) = 0$. Therefore, if $l = 1$ and $r_v(G_1) = 0$, we have $\sum_1^2 r_v(G_i) = 1$. Hence, by (2), $r_v(G) = 1$, that is, $\mathrm{mr}(G) = \sum_1^2 \mathrm{mr}(G_i - v) + 1 = \mathrm{mr}(G_1 - v) + 1 = \mathrm{mr}(G_1) + 1$, since $r_v(G_1) = 0$.

On the other hand, if either $l > 1$ or $r_v(G_1) > 0$, by (2) we have $r_v(G) = 2$, that is, $\mathrm{mr}(G) = \sum_1^{l+1} \mathrm{mr}(G_i - v) + 2 = \mathrm{mr}(G_1 - v) + 2 = \mathrm{mr}(G) + 2 - r_v(G_1)$. □

We now turn our attention to edge-sums of graphs and use the above analysis pertaining to vertex-sums to obtain analogous results for edge-sums. Let $G_1$ and $G_2$ be disjoint undirected graphs, and let $v_1$ and $v_2$ be vertices of $G_1$ and $G_2$ respectively. If we connect $G_1$ and $G_2$ by adding the edge $e = \{v_1, v_2\}$, the resulting graph $G$ is called *edge-sum* of $G_1$ and $G_2$, and is denoted by $G = G_1 \underset{e}{+} G_2$.

**Theorem 2.6** *Let $G = G_1 \underset{e}{+} G_2$, with $e = \{v_1, v_2\}$. Then*

$$\mathrm{mr}(G) = \begin{cases} \mathrm{mr}(G_1) + \mathrm{mr}(G_2) & \text{if } r_{v_i}(G_i) = 2 \text{ for at least one } i; \\ \mathrm{mr}(G_1) + \mathrm{mr}(G_2) + 1 & \text{otherwise.} \end{cases}$$

**Proof** Denote by $H$ the graph obtained by appending the edge $\{v_1, v_2\}$ to $G_1$. Let us assume $r_{v_1}(G_1) = 2$, so that, by Lemma 2.5, we have $r_{v_1}(H) = 2$ and so $\mathrm{mr}(H) = \mathrm{mr}(G_1)$. We now consider $G$ as vertex sum at $v_2$ of $H$ and $G_2$. Note that $r_{v_2}(H) = \mathrm{mr}(H) - \mathrm{mr}(G_1) = 0$, so, with regard to (2), we have

$$\begin{aligned} r_{v_2}(G) &= \min\{r_{v_2}(H) + r_{v_2}(G_2), 2\} = r_{v_2}(G_2) \\ &= \mathrm{mr}(G_2) - \mathrm{mr}(G_2 - v_2). \end{aligned} \tag{4}$$

On the other hand,

$$r_{v_2}(G) = \mathrm{mr}(G) - \mathrm{mr}(G - v_2) = \mathrm{mr}(G) - \mathrm{mr}(G_1) - \mathrm{mr}(G_2 - v_2). \tag{5}$$

By comparing (4) and (5) we obtain $\mathrm{mr}(G) = \mathrm{mr}(G_1) + \mathrm{mr}(G_2)$.

Let now consider the case $r_{v_i}(G_i) \leqslant 1$ for each $i$. By Lemma 2.5 we have in any case $\mathrm{mr}(H) > \mathrm{mr}(G_1)$, that is, $r_{v_2}(H) > 0$. Thus, in this case

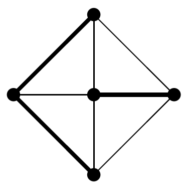
$$\begin{aligned} r_{v_2}(G) &= \min\{r_{v_2}(H) + r_{v_2}(G_2), 2\} \\ &> r_{v_2}(G_2) = \mathrm{mr}(G_2) - \mathrm{mr}(G_2 - v_2), \end{aligned} \tag{6}$$

since $r_{v_2}(G_2) \leqslant 1$. By comparing (6) and (5), we now have $\mathrm{mr}(G) > \mathrm{mr}(G_1) + \mathrm{mr}(G_2)$. Finally, since $\mathrm{mr}(G) \leqslant \mathrm{mr}(G_1) + \mathrm{mr}(G_2) + 1$ (cf. [N, Prop 2.1]), we obtain the desired conclusion. □

## 3 The Relationship between Maximum Multiplicity, Path Cover Number and $\Delta$

In [JLD99] Johnson and Leal Duarte showed that for trees, $\Delta(T) = P(T) = M(T)$. We consider the relationship between these parameters for graphs in general. It is easy to find an example in which $\Delta(G) < P(G) < M(G)$, for instance, $W_5$, the wheel on five vertices, Figure 2, which has $\Delta(W_5) = -1$, $P(W_5) = 2$, and $M(W_5) = 3$, since $\text{mr}(W_5) = 2$ by [BL]. A larger discrepancy between $P$ and $M$ may be obtained by considering $K_n$, the complete graph on $n$ vertices. If $n$ is even then $P(K_n) = n/2$, but $M(K_n) = n - 1$ (because $\text{mr}(K_n) = 1$).

Figure 2: $W_5$, showing a minimal path cover



The following question naturally arises: Is it true for any graph $G$ that $\Delta(G) \leqslant P(G) \leqslant M(G)$? The proof in [JLD99] establishes $\Delta(G) \leqslant M(G)$ for any graph, since it utilizes interlacing inequalities and does not rely on $G$ being a tree. The next theorem establishes the first inequality.

**Theorem 3.1** *Let $G$ be a graph.*

*i. If $e$ is an edge of $G$, then $\Delta(G) \leqslant \Delta(G - e)$;*

*ii. $\Delta(G) \leqslant P(G)$.*

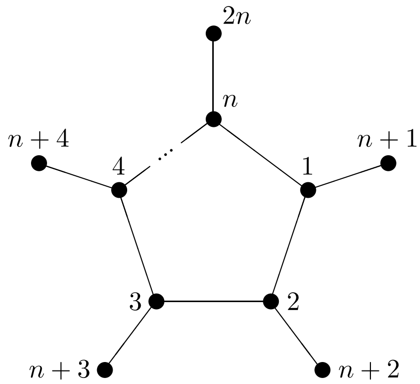
**Proof**

i. Recall that $\Delta(G) = \max\{p - q$: there are $q$ vertices of $G$ whose deletion leaves $p$ paths$\}$. In $G$ choose a set $Q$ of $q$ vertices leaving $p$ paths such that $p - q = \Delta(G)$. If $e$ is incident with a vertex in $Q$, then deletion of the vertices in $Q$ leaves the same $p$ paths in $G - e$ and thus (by maximality) $p - q \leqslant \Delta(G - e)$. If $e$ is not incident with a vertex in $Q$, then $e$ is in one of the paths and the removal of $e$ creates an additional path, so $p + 1 - q \leqslant \Delta(G - e)$. In either case, $\Delta(G) \leqslant \Delta(G - e)$.

ii. Choose a minimal path cover $\Psi$ for $G$. By i., $\Delta(G) \leqslant \Delta(\Psi)$. $\Psi$ is a disjoint union of trees, so $\Delta(\Psi) = P(\Psi) = P(G)$, by choice of $\Psi$. □

The next result answers the remaining question in the negative, that is, we exhibit graphs with $P(G) > M(G)$. For any $n \geqslant 3$, the *$n$-sun* $H_n$ is the corona graph of an $n$-cycle, namely, the graph on $2n$ vertices obtained by appending a leaf on each vertex of an $n$-cycle.

Figure 3: The $n$-sun $H_n$




**Proposition 3.2** *Let $H_n$ be the $n$-sun on $2n$ vertices. Then*

*i.* $P(H_n) = \lceil \frac{n}{2} \rceil$, $n \geqslant 3$;

*ii.* $\mathrm{mr}(H_3) = 4$;

*iii.* $\mathrm{mr}(H_n) = 2n - \lfloor \frac{n}{2} \rfloor$, $n > 3$.

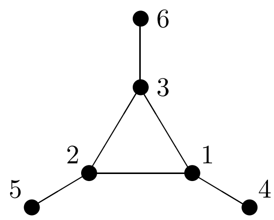
*In particular, if $n > 3$ is odd, $P(H_n) > M(H_n)$.*

**Proof**

i. Since $H_n$ has exactly $n$ leaves, and each path can cover at most two of them, we have $P(H_n) \geqslant \lceil \frac{n}{2} \rceil$. A path cover of cardinality $\lceil \frac{n}{2} \rceil$ is easily obtained by connecting with a path pairs of adjacent leaves.

ii. Let $G = H_3 - (6)$ (see Figure 4), $G' = G - (3)$. Note that $G' = P_3$, so

Figure 4: The 3-sun $H_3$



that $\mathrm{mr}(G') = 3$. Moreover $\mathrm{mr}(G) = 3$, since $G$ contains $P_3$ as an induced subgraph, but $G \neq P_4$. We are in a position to apply Theorem 2.6 and obtain that $\mathrm{mr}(H_3) = \mathrm{mr}(G \underset{\{3,6\}}{+} (6)) = \mathrm{mr}(G) + 0 + 1 = 4$.

iii. Let $A$ be any matrix with $G(A) = H_n$, and consider the diagonal entries corresponding to the $n$ leaves. Suppose $h$ of these entries are nonzero. Therefore, by reordering and scaling rows and columns, we can assume that $A$ is in the form

$$A = \begin{bmatrix} A_{11} & A_{12} & I_h & \mathbf{O} \\ A_{21} & A_{22} & \mathbf{O} & I_{n-h} \\ I_h & \mathbf{O} & I_h & \mathbf{O} \\ \mathbf{O} & I_{n-h} & \mathbf{O} & \mathbf{O} \end{bmatrix}. \tag{7}$$

By performing suitable sums on rows and columns, it is easy to see that

$$\begin{aligned} \operatorname{rank} A &= \operatorname{rank} I_h + 2 \operatorname{rank} I_{n-h} + \operatorname{rank}(A_{11} - I_h) & (8) \\ &\geqslant h + 2(n-h) + \mathrm{mr}(G(A_{11})). & (9) \end{aligned}$$

Case I: $h = n$. We have $G(A_{11}) = C_n$, the $n$-cycle, hence $\mathrm{mr}(G(A_{11})) = n - 2$ [N], and by (9) we obtain $\operatorname{rank} A \geqslant 2n - 2 \geqslant 2n - \lfloor \frac{n}{2} \rfloor$.

Case II: $h < n$. Here $G(A_{11})$ is disjoint union of, say, $k$ paths. So $\mathrm{mr}(G(A_{11})) = h - k$. By (9), $\operatorname{rank} A \geqslant 2n - k$. Note that the $k$ paths in $G(A_{11})$ are obtained by deleting exactly $n - h$ vertices from the $n$-cycle. Therefore $k \leqslant n - h$. This inequality, together with the obvious $k \leqslant h$, yields $k \leqslant \lfloor \frac{n}{2} \rfloor$, and finally $\operatorname{rank} A \geqslant 2n - \lfloor \frac{n}{2} \rfloor$.

A matrix with graph $H_n$ and rank $2n - \lfloor \frac{n}{2} \rfloor$ is obtained by defining

$$A = \begin{bmatrix} C + D & I_n \\ I_n & D \end{bmatrix},$$

where $D = \mathrm{diag}(0, 1, 0, 1, \ldots)$ and

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \ddots & & 0 \\ 0 & 1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & & \ddots & 1 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

By reordering the vertices and writing $A$ as in (7), a simple check proves that $h = \lfloor \frac{n}{2} \rfloor$ and $A_{11} = I_h$. By (8) we now obtain $\operatorname{rank} A = \lfloor \frac{n}{2} \rfloor + 2\lceil \frac{n}{2} \rceil = 2n - \lfloor \frac{n}{2} \rfloor$.

□

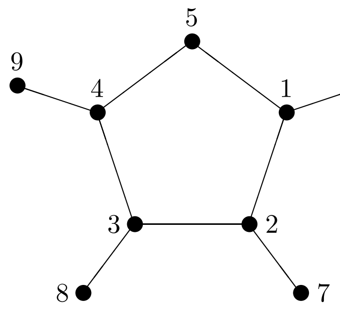
Although in general it is not difficult to find a graph, such as the wheel $W_5$, in which $P(G) < M(G)$, we shall see that this requires adjacent cycles. In fact, we will show that any graph built by edge-sums from graphs all of whose induced subgraphs have $P(G) \geqslant M(G)$ will also have this property. Define a graph to be *non-deficient* if for all induced subgraphs $H$ of $G$, $\mathrm{mr}(H) + P(H) \geqslant |H|$, or equivalently $P(H) \geqslant M(H)$. A vertex $v$ is a *terminal* vertex in $G$ if $v$ is the end point of a path in some minimum path cover of $G$. We first obtain some bounds on path cover number analogous to those established for minimum rank.

**Lemma 3.3** *For any vertex $v$ of $G$, $P(G) - 1 \leqslant P(G - v) \leqslant P(G) + 1$. If $v$ is terminal in $G$ then $P(G - v) \leqslant P(G)$.*

**Proof** If there is a minimum path cover in which $v$ is an endpoint then this cover with $v$ deleted provides a path cover with no more than $P(G)$ paths, so $P(G - v) \leqslant P(G)$. Otherwise when this cover is considered in $G - v$, one path will split into two and so $P(G - v) \leqslant P(G) + 1$.
For any minimum path cover of $P(G - v)$ this path cover together with $v$ is a path cover of $G$, so $P(G) \leqslant P(G - v) + 1$. □

**Example 3.4** Although the second statement in Lemma 3.3 guarantees that deleting any terminal vertex implies $P(G - v) \leqslant P(G)$, the converse is false, as can be seen by considering vertex 5 in $G = H_5 - (10)$ (see Figure 5). The paths $(6, 1, 5, 4, 9)$ and $(7, 2, 3, 8)$ are the only minimum path cover of $G$, so that 5 is not terminal. Moreover the paths $(6, 1, 2, 7)$ and $(8, 3, 4, 9)$ are (the only) minimum path cover of $G - (5)$. Thus, $P(G - v) = P(G)$.

Figure 5: The graph $G = H_5 - (10)$




**Lemma 3.5** *Let $G = G_1 \underset{e}{+} G_2$ with $e = \{v_1, v_2\}$. Then*

$$P(G) = \begin{cases} P(G_1) + P(G_2) - 1 & \text{if and only if } v_i \text{ is terminal in } G_i,\ \forall i = 1, 2; \\ P(G_1) + P(G_2) & \text{otherwise.} \end{cases}$$

**Proof** The union of path covers for $G_1$ and $G_2$ is a path cover for $G$, so by using minimal path covers for $G_i$, $P(G) \leqslant P(G_1) + P(G_2)$.

Given a minimal path cover $\Psi$ for $G$ (so $|\Psi| = P(G)$), we obtain path covers $\Psi_i$ for $G_i$ as pieces of this. Clearly the number of paths in the union of these covers is either the same number of paths as the original or one more, that is, $P(G_1) + P(G_2) \leqslant |\Psi_1| + |\Psi_2| \leqslant P(G) + 1$.

If $P(G_1)+P(G_2) = P(G)+1$, then the edge $e$ appeared in $\Psi$, so $v_i$ is terminal in $\Psi_i$ for both $i = 1, 2$. Since $P(G_1)+P(G_2) \leqslant |\Psi_1|+|\Psi_2| \leqslant P(G)+1$, we have $P(G_1) + P(G_2) = |\Psi_1| + |\Psi_2| = P(G) + 1$. Since $|P(G_i)| \leqslant |\Psi_i|$, for $i = 1, 2$, necessarily $|P(G_i)| = |\Psi_i|$, that is, the covers $\Psi_1$ and $\Psi_2$ of $G_1$ and $G_2$ produced from the cover $\Psi$ of $G$ must be minimal, and $v_i$ was the end point of a path in $\Psi_i$. Thus $P(G_1) + P(G_2) - 1 = P(G)$ implies $v_i$ is terminal in $G_i$ for both $i = 1, 2$.

If for both $i = 1, 2$ $v_i$ is terminal in $G_i$, then a path cover of size $P(G_1) + P(G_2) - 1$ for $G$ is obtained from minimal path covers for $G_1$ and $G_2$ in which the vertices are terminal by joining the path ending in $v_1$ to the path ending in $v_2$ by edge $e$, so in this case $P(G) = P(G_1) + P(G_2) - 1$. □

**Theorem 3.6** *Let $G = G_1 \underset{e}{+} G_2$ with $e = \{v_1, v_2\}$. If both $G_1$ and $G_2$ are non-deficient then $G$ is non-deficient. Thus,* $\text{mr}(G) + P(G) \geqslant |G|$*, or equivalently,* $P(G) \geqslant M(G)$.

**Proof** Let $H$ be an induced subgraph of $G$. Let $H_i$ be the subgraph induced by $V(H) \cap V(G_i)$. If for some $i$, $v_i$ is not in $H_i$ then $H$ is the disjoint union of $H_1$ and $H_2$ and the result is clear. So assume $v_i$ in $H_i$ for $i = 1, 2$, and so $H = H_1 \underset{e}{+} H_2$. By Theorem 2.6, we have

case 1. $\text{mr}(H) = \text{mr}(H_1) + \text{mr}(H_2) + 1$ or

case 2. $\text{mr}(H) = \text{mr}(H_1) + \text{mr}(H_2)$.

By Lemma 3.5, we have

case a. $P(H) = P(H_1) + P(H_2)$ or

case b. $P(H) = P(H_1) + P(H_2) - 1$.

In case (1),

$$\begin{aligned} \text{mr}(H) + P(H) &\geqslant \text{mr}(H_1) + \text{mr}(H_2) + 1 + P(H_1) + P(H_2) - 1 \\ &\geqslant |H_1| + |H_2| \\ &= |H|. \end{aligned}$$

In case (a),

$$\begin{aligned} \text{mr}(H) + P(H) &\geqslant \text{mr}(H_1) + \text{mr}(H_2) + P(H_1) + P(H_2) \\ &\geqslant |H_1| + |H_2| \\ &= |H|. \end{aligned}$$

Finally suppose both case (2) and case (b) hold. We know from Theorem 2.6 that for some $i$ (say $i = 1$), $r_{v_1}(H_1) = \text{mr}(H_1) - \text{mr}(H_1 - v_1) = 2$. From Lemma 3.5, both $v_1$ and $v_2$ are terminal in $H_1$ and $H_2$, respectively. Then by Lemma 3.3, $P(H_1 - v_1) \leqslant P(H_1)$. Since $H_1$ is non-deficient, $|H_1| - 1 = |H_1 - v_1| \leqslant \text{mr}(H_1 - v_1) + P(H_1 - v_1) \leqslant \text{mr}(H_1) - 2 + P(H_1)$. Thus $|H_1| + 1 \leqslant \text{mr}(H_1) + P(H_1)$. Also, $|H_2| \leqslant \text{mr}(H_2) + P(H_2)$. Therefore, $\text{mr}(H) + P(H) = \text{mr}(H_1) + \text{mr}(H_2) + P(H_1) + P(H_2) - 1 = \text{mr}(H_1) + P(H_1) + \text{mr}(H_2) + P(H_2) - 1 \geqslant |H_1| + 1 + |H_2| - 1 = |H_1| + |H_2| = |H|$. □

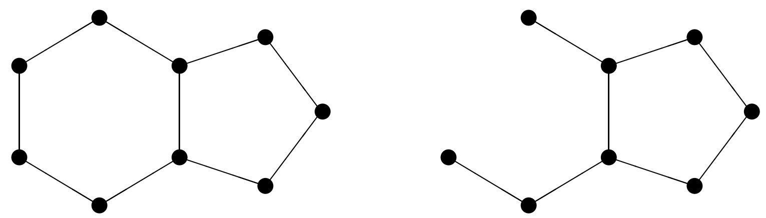
For any tree, or more generally, forest, $\text{mr}(T) + P(T) = |T|$ [JLD99], and any proper induced subgraph of a forest is a forest, so a forest (or tree) is non-deficient.

**Example 3.7**

i. For any cycle $C_n$, $\text{mr}(C_n) + P(C_n) = n = |C_n|$, because $P(C_n) = 2$ and $\text{mr}(C_n) = n - 2$ [N].

ii. Let $C_{m,n}$, denote the double cycle which consists of one cycle of length $m$ and one cycle of length $n$ sharing one common edge (see the graph on the left in Figure 6). Then $P(C_{m,n}) = 2$ (use one end of the shared edge as a path and all remaining vertices as a path), and $\text{mr}(C_{m,n}) = n-2$ as for the cycle. Any proper connected induced subgraph $H$ of $C_{m,n}$ is a cycle with at most two paths adjoined or is a tree. If $H$ contains a cycle, then $P(H) = 2$, while $\text{mr}(H) \leqslant |H| - 2$, since $H$ is not a path. But $H$ is non-deficient by item (i.) above and Theorem 3.6, so $\text{mr}(H) \geqslant |H| - P(H) = |H| - 2$.

Figure 6: The double cycle $C_{6,5}$ and induced subgraph



## 4 "Special" Vertices

It is clear that vertices $v$ having the property that $\text{mr}(G) - \text{mr}(G - v) = 2$, or equivalently, $M(G - v) = M(G) + 1$, have played a crucial role in this discussion. We call such a $v$ a *rank-strong* vertex in $G$. In much of the analysis of multiplicities of eigenvalues of trees, various kinds of "special" vertices with this and additional properties have been exploited. One such kind of vertex is what Nylen has called an appropriate vertex. A vertex $v$ is *appropriate* in $G$ (in the sense of [N]) if its deletion from $G$ has at least 2 components that are paths joined at the end to the deleted vertex. Such vertices are exploited in [N] to compute $\mathrm{mr}(T)$ for $T$ a tree. Although Nylen defined and used appropriate vertices only for trees, in any graph $G$ any appropriate vertex is a rank-strong vertex (see Proposition 4.1). Wei and Weng [WW] call a vertex $v$ of a tree $T$ *typical* if $v$ has at least two neighbors of degree less than or equal to 2, and use typical vertices to calculate $\mathrm{mr}(T)$. As noted in [WW], every appropriate vertex is typical but not vice versa. Although in any tree a typical vertex (in the sense of [WW]) is a rank-strong vertex (see Corollary 4.3), this is not true in general: consider an $n$-cycle, where every vertex is typical but not rank-strong. Not every rank-strong vertex is typical, even in a tree (see Example 4.4). Before justifying these remarks through a series of propositions, it is worth mentioning that leaves are never rank-strong vertices, since, by appending a leaf to a graph, the minimal rank cannot increase by more than one. On the other hand, if $v$ is rank-strong in a graph $G_1$, then (Theorem 2.3) $v$ remains rank-strong in any graph obtained by doing a vertex-sum on $v$, i.e., for any graphs $G_2, \ldots, G_h$ and $G$ such that $G$ is vertex-sum at $v$ of $G_1, \ldots, G_h$, $v$ is rank-strong in $G$.

**Proposition 4.1** *Any appropriate vertex of a graph is rank-strong.*

**Proof** $G$ is vertex-sum of graphs $G_1, \ldots, G_h$, in which at least two components are paths. Note that if $v$ is an extreme vertex of a path $P$, then $r_v(P) = 1$. By applying Theorem 2.3 we have $r_v(G) = 2$. □

In order to obtain a similar result for typical vertices of a tree, we first notice that, since, for a tree, $P(T) = M(T)$, we have

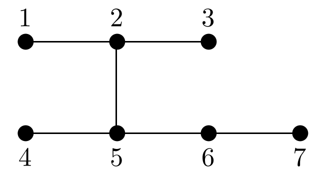
**Proposition 4.2** *A vertex $v$ in a tree $T$ is rank-strong if and only if $P(T-v) = P(T) + 1$.*

**Corollary 4.3** *Any typical vertex of a tree is rank-strong.*

**Proof** By Proposition 4.2, it suffices to show that, in a tree, the removal of a typical vertex $v$ increases the path cover number. Let $w_1$ and $w_2$ be two low degree neighbors of $v$ guaranteed by the definition of typical, and let $T - v$ have components $T_1$, $T_2$, $\ldots$, $T_k$. Since $T$ is a tree, $w_1$ and $w_2$ must be in distinct components, say $T_1$ and $T_2$. Moreover $w_i$ $(i = 1, 2)$ must be terminal in $T_i$, and since the union of minimal path covers of the $T_i$'s is a minimal path cover of $T - v$, we can obtain a path cover for $G$ with $P(T - v) - 1$ paths by joining the path ending at $w_1$ to $v$ to the path ending at $w_2$. Thus $P(T) \leqslant P(T - v) - 1$ and equality follows from Lemma 3.3. □

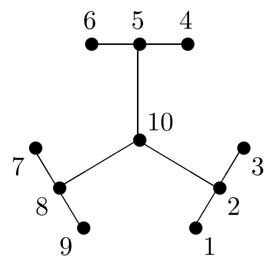
**Example 4.4** The converse of Corollary 4.3 is not true. Let $T$ be the double-path in Figure 7 on the next page. Vertex 6 is not a typical vertex. However 6 is a rank-strong vertex, since $P(T) = 2$, $P(T - 6) = 3$.

Figure 7: A tree with a rank-strong vertex that is not typical




For a given matrix $A$ and eigenvalue $\lambda$ of $A$ with $\text{mult}_A(\lambda) > 1$, we will call a vertex $v$ of $G(A)$ a *Parter-Wiener* vertex for $\lambda$ if 1) $\lambda$ is an eigenvalue of at least 3 irreducible components of $A(v)$ and 2) $\text{mult}_{A(v)}(\lambda) = \text{mult}_A(\lambda) + 1$. Such a vertex $v$ has been called a *Parter vertex* in [JLD02] and [JDSSW], and a *Wiener vertex* in [BFgen] and [BFconj]. In [P] and [W] it is established implicitly that if $T = G(A)$ is a tree, then $T$ must have a Parter-Wiener vertex for any multiple eigenvalue $\lambda$ of $A$. However, the $n$-cycle $C_n$ has no Parter-Wiener vertices and no rank-strong vertices even though there is a matrix $A$ with $G(A) = C_n$ having an eigenvalue of multiplicity 2. Since, as noted in [JDSSW], for each vertex $v$ of a tree $T$, whose degree is larger than or equal to 3, it is possible to construct a matrix $A$ with $v$ as Parter-Wiener vertex for an eigenvalue of $A$, we can easily construct a Parter-Wiener vertex for a matrix $A$ which is not a rank-strong vertex for $T = G(A)$. Consider, for instance, the tree shown in Figure 8. By Proposition 4.2, vertex 10 is a not rank-strong vertex, but it has degree larger than 3.

Figure 8: Vertex 10 is a not rank-strong vertex, but can be a Parter-Wiener




However, by comparing the definitions of a rank-strong vertex and a Parter-Wiener vertex, we have the following as an immediate consequence.

**Proposition 4.5** *Let $G$ be a graph. If $v$ is a Parter-Wiener vertex of a matrix $A$ and eigenvalue $\lambda$ of $A$ with* $\text{mult}_A(\lambda) = M(G)$*, then $v$ is a rank-strong vertex of $G$.*

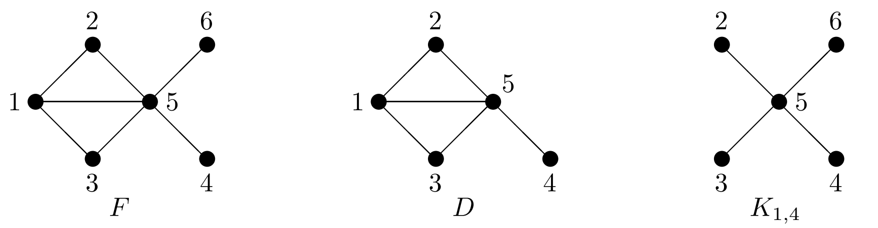
So the idea of a rank-strong vertex appears to generalize (in a way that is useful to the study of $\text{mr}(G)$) the ideas of an appropriate vertex, a typical vertex in a tree, and a Parter-Wiener vertex for an eigenvalue of maximum multiplicity.

## 5 Induced Subgraphs

A useful property of minimal rank is that it behaves monotonically on induced subgraphs, that is, if $H$ is an induced subgraph of $G$ then $\text{mr}(H) \leqslant \text{mr}(G)$. However, that is not true for the other parameters we have discussed, as the following example shows.

**Example 5.1** Let $F$ be the first graph shown in Figure 9. Then $\text{mr}(F) = 4$ because $F$ is the edge-sum of the dart $D$ and vertex 6. Further, $\text{mr}(D) = 3$ by [BL], vertex 5 is not rank-strong in $D$, and vertex 6 is a leaf. Thus $M(F) = 2$. Finally, a simple check shows that $P(F) = 2$ and $\Delta(F) = 2$. On the other hand, consider the subgraph $K_{1,4}$ of $F$ induced by $2, 3, 4, 5, 6$. Since $K_{1,4}$ is a star, $\text{mr}(K_{1,4}) = 2$. Thus $M(K_{1,4}) = 3 > 2 = M(F)$. Since $K_{1,4}$ is a tree, $M(K_{1,4}) = P(K_{1,4}) = \Delta(K_{1,4})$ and so $P(K_{1,4}) > P(F)$ and $\Delta(K_{1,4}) > \Delta(F)$.

Figure 9: The graph $F$ and induced subgraphs $D$ and $K_{1,4}$




In the case of a tree, $\Delta$, $P$ and $M$ do behave monotonically on connected induced subgraphs, since the path cover number of a connected induced subgraph is always smaller than or equal to the path cover number of the whole graph. We summarize this fact as follows.

**Proposition 5.2** *If $T$ is a tree and $H$ is a connected induced subgraph of $T$ then $P(H) \leqslant P(T)$. Therefore $M(H) \leqslant M(T)$, and $\Delta(H) \leqslant \Delta(T)$.*

Note that the statement of Proposition 5.2 can be false if the requirement that $H$ be connected is removed.

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