Families of graphs with maximum nullity equal to zero-forcing number

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Abstract

The maximum nullity of a simple graph $G$, denoted $M(G)$, is the largest possible nullity over all symmetric real matrices whose $ij$th entry is nonzero exactly when $\{i, j\}$ is an edge in $G$ for $i \neq j$, and the $ii$th entry is any real number. The zero-forcing number of a simple graph $G$, denoted $Z(G)$, is the minimum number of blue vertices needed to force all vertices of the graph blue by applying the color change rule. This research is motivated by the longstanding question of characterizing graphs $G$ for which $M(G) = Z(G)$. The following conjecture was proposed at the 2017 AIM workshop Zero forcing and its applications: If $G$ is a bipartite 3-semiregular graph, then $M(G) = Z(G)$. A counterexample was found by J. C.-H. Lin but questions remained as to which bipartite 3-semiregular graphs have $M(G) = Z(G)$. We use various tools to find bipartite families of graphs with regularity properties for which the maximum nullity is equal to the zero-forcing number; most are bipartite 3-semiregular. In particular, we use the techniques of twinning and vertex sums to form new families of graphs for which $M(G) = Z(G)$ and we additionally establish $M(G) = Z(G)$ for certain Generalized Petersen graphs.

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1 Introduction

Let $V$ be a finite nonempty set. A graph $G = (V, E)$ is a pair of sets such that $E$ is a set of two elements subsets of $V$. The elements of $V$ are called vertices and the elements of $E$ are called edges. The vertex set of a graph $G$ is often denoted by $V(G)$ and the edge set by $E(G)$. The order of $G$ is the cardinality of $V(G)$ and the size of $G$ is the cardinality of $E(G)$.

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An edge \( \{u,v\} \) is usually written as \( uv \). Vertices \( u \) and \( v \) are adjacent in \( G \) if \( uv \in E(G) \). A vertex \( u \) is a neighbor of \( v \) if \( uv \in E(G) \). The neighborhood of \( v \), denoted by \( N(v) \), is the set of neighbors of \( v \).

For a graph \( G \), the set of symmetric matrices of \( G \) over \( \mathbb{R} \), denoted by \( S(G) \), is the set of real symmetric matrices \( A = [a_{ij}] \) such that \( a_{ij} \) is non-zero if \( ij \in E(G) \), \( a_{ij} \) is any real number if \( i = j \), and \( a_{ij} = 0 \) otherwise. The minimum rank of \( G \) is \( mr(G) = \min\{\text{rank}(A) \mid A \in S(G) \} \). The maximum nullity of \( G \) is defined as \( M(G) = \max\{\text{null}(A) \mid A \in S(G) \} \). Observe that \( mr(G) + M(G) = |V(G)| \) where \( | \cdot | \) denotes cardinality. The adjacency matrix of \( G \) is \( A(G) = [a_{ij}] \) where \( a_{ij} = 1 \) if \( ij \in E(G) \) and \( a_{ij} = 0 \) otherwise.

In order to introduce zero-forcing we will define the color change rule as follows: Suppose a graph \( G \) has every vertex colored either blue or white, and \( b \) is a blue vertex. If \( b \) has exactly one white neighbor, \( w \), then we change the color of \( w \) to blue. We say that \( b \) forces \( w \), and this can be denoted by \( b \rightarrow w \). Let \( S \subseteq V(G) \). The final coloring of \( S \) is the result of initially coloring every vertex in \( S \) blue and every vertex in \( V(G) \setminus S \) white, and then applying the color-change rule until no more color changes can be made. Note that the order in which forces occur does not affect the final coloring of \( G \). The set \( S \) is called a zero-forcing set if the final coloring of \( S \) is all blue. The zero-forcing number of a graph \( G \) is \( Z(G) = \min\{|S| \mid S \text{ is a zero-forcing set of } G \} \). It is well known from \([1]\) that \( M(G) \leq Z(G) \). This paper addresses the longstanding question of determining graphs \( G \) for which \( M(G) = Z(G) \) (see \([1, \text{Question 1}]\)).

The degree of \( v \), \( \deg(v) \), is the number of edges incident to \( v \). Note that \( \deg(v) = |N(v)| \). A vertex with degree equal to 1 is called a leaf. The minimum degree of \( G \) is \( \delta(G) = \min\{\deg(v) \mid v \in V(G) \} \) and the maximum degree of \( G \) is \( \Delta(G) = \max\{\deg(v) \mid v \in V(G) \} \). A graph \( G \) is regular if \( \Delta(G) = \delta(G) \). We call \( G \) cubic if \( \Delta(G) = 3 = \delta(G) \). A graph \( G \) is bipartite if \( V(G) \) can be partitioned into two sets \( X \) and \( Y \) such that \( N(x) \subseteq Y \) and \( N(y) \subseteq X \) for \( x \in X \) and \( y \in Y \); the partition of edges can be denoted by \( G(X,Y) \). Let \( G = G(X,Y) \) be a bipartite graph. We say \( G \) is \( k \)-semiregular if the degree of \( x \) is \( k \) for all \( x \in X \). We say \( G \) is \((k,\ell)\)-biregular if the degree of \( x \) is \( k \) for all \( x \in X \) and the degree of \( y \) is \( \ell \) for all \( y \in Y \).

At the 2017 American Institute of Mathematics workshop Zero forcing and its applications, it was conjectured that \( M(G) = Z(G) \) if \( G \) is a bipartite 3-semiregular graph \([2]\). It is known that all bipartite 1-semiregular and bipartite 2-semiregular graphs satisfy \( M(G) = Z(G) \), but not all bipartite graphs have \( M(G) = Z(G) \). A counterexample to the conjecture was found by J.C.-H. Lin \([8]\) (see Example 1.1), but questions remain as to which bipartite 3-semiregular graphs have \( M(G) = Z(G) \). In this paper, we construct bipartite families of graphs with regularity properties for which the maximum nullity is equal to the zero-forcing number. In Section 2 we establish that \( M(G) = Z(G) \) for many Generalized Petersen graphs (these graphs are all cubic and some are bipartite). In Section 3 we develop expansion techniques that preserve \( M(G) = Z(G) \) and apply them to families of graphs in Section 4. In particular, we use twinning and vertex sums to form new families of bipartite graphs for which \( M(G) = Z(G) \); some of these are 3-semiregular. In Section 5 we show that \( M(G) = Z(G) \) for two well-known cubic bipartite graphs.

**Example 1.1.** Let \( L \) be the bipartite 3-semiregular graph shown in Figure 1.1. The path cover number of a graph \( G \), denoted by \( P(G) \), is the minimum number of induced paths
needed to include all vertices of $G$. A graph is outerplanar if there is a drawing of the graph with no crossings and all vertices on the infinite face. Sinkovic [10] showed that $M(G) \leq P(G)$ for an outerplanar graph $G$. Since $L$ is outerplanar and $P(L) = 3$, $M(L) \leq 3$. By use of software [6] it is straightforward to verify that $Z(L) = 4$.

![Figure 1.1: Lin’s example $L$ of a bipartite 3-semiregular graph with $M(L) < Z(L)$](image)

A graph $H$ is a subgraph of graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $S \subseteq V(G)$ and $E[S] = \{uv \in E(G) \mid u, v \in S\}$. The subgraph of $G$ induced by $S$ is $G[S] = (S, E[S])$. That is, an induced graph is one obtained by deleting vertices and incident edges.

A graph $G$ is a path on $n$ vertices, denoted by $P_n$, if $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_n\}$. A graph $G$ is a cycle on $n$ vertices, denoted by $C_n$, if $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. A complete graph on $n$ vertices, denoted by $K_n$, is a graph of order $n$ with all possible edges between its vertices. A complete bipartite graph is a bipartite graph with all possible edges between the two parts and is denoted by $K_{n,m}$ where $n$ and $m$ are the orders of the two parts.

The union of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $(V_1 \cup V_2, E_1 \cup E_2)$, denoted $G_1 \cup G_2$. If $V_1 \cap V_2 \neq \emptyset$, then the intersection of $G_1$ and $G_2$ is the graph $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$. Suppose that $G_1$ and $G_2$ are graphs such that $V(G_1) \cap V(G_2) = \{v\}$ (and $V(G_i) \neq \{v\}$ for $i = 1, 2$). Then $G_1 \cup G_2$ is called the vertex sum of $G_1$ and $G_2$ at $v$ and is denoted $G_1 \oplus G_2$. A graph $G = (V, E)$ is connected if for each pair of vertices $u, v \in V$ there exists a path from $u$ to $v$. A (connected) component of $G$ is a maximal connected subgraph of $G$. A vertex $v$ is a cut vertex of $G$ if the number of components of $G$ is less than the number of components of $G - v$, where $G - v = G[V \setminus \{v\}]$. Observe that $v$ is a cut vertex of $G_1 \oplus G_2$, and if $v$ is a cut vertex of $G$, then $G$ can be expressed as a vertex sum at $v$.

## 2 Generalized Petersen Graphs

In this section, we present results on the zero-forcing number and maximum nullity of the Generalized Petersen (abbreviated GP) graphs, including exact values for a few subfamilies of GP graphs. We first define the aforementioned family using notation similar to that
in [11]. For $n \geq 3$ and $0 < k \leq \lfloor \frac{n-1}{2} \rfloor$, the Generalized Petersen (GP) graph $P(n,k)$ is the graph with vertex set $V = \{u_0, u_1, u_2, \ldots, u_{n-1}, v_0, v_1, v_2, \ldots, v_{n-1}\}$ and edge set $E = \{u_i u_{i+1} \text{ for all } 0 \leq i \leq n-1, u_i v_i \text{ for all } 0 \leq i \leq n-1, v_j v_{j+k} \text{ for all } 0 \leq j \leq n-1\}$ where subscripts are taken modulo $n$.

![Figure 2.1: A Generalized Petersen graph (shown with a zero-forcing set)](image)

The graph $P(n,k)$ is a cubic graph consisting of an outside cycle on vertices $u_0, \ldots, u_{n-1}$ with a perfect matching connecting this cycle to one or more inner cycles on vertices $v_0, \ldots, v_{n-1}$ where the vertices on the inner and outer cycles are ordered counterclockwise (see Figure 2.1). In the literature, the term Generalized Petersen graph often requires that $n$ and $k$ be relatively prime. However, for the purposes of this paper we do not require this condition. Note that in the above definition $k$ is restricted as $0 < k \leq \lfloor \frac{n-1}{2} \rfloor$. This is because the symmetry of the GP graphs gives us that for $\lceil \frac{n-1}{2} \rceil < \ell < n$, $P(n, \ell)$ is isomorphic to $P(n, n - \ell)$, and $0 < n - \ell < \lfloor \frac{n-1}{2} \rfloor$. Some GP graphs are known by different names such as the $n$-Prism, which is $P(n,1)$, and the well-known Petersen graph $P(5,2)$. It is known that $M(P(n,1)) = Z(P(n,1)) = 4$ for $n \geq 4$, $M(P(3,1)) = Z(P(3,1)) = 4$, and $M(P(5,2)) = Z(P(5,2)) = 5$ [1].

**Remark 2.1.** A GP graph is bipartite if and only if $n$ is even and $k$ is odd. This is because $P(n,k)$ contains an odd cycle if $n$ is odd or if $k$ is even, and a graph is bipartite if and only if it does not contain an odd cycle.

**Theorem 2.2.** For any $n \geq 3$, $Z(P(n,k)) \leq 2k + 2$.

**Proof.** Let $S_0 = \{v_0, u_0, u_1, u_2, \ldots, u_{2k-1}, v_{2k-1}\}$, $S_1 = S_0 \cup \{v_1, v_2, \ldots, v_{2k-2}\}$, and $S_i = S_{i-1} \cup \{v_{n-i+1}, u_{n-i+1}\}$ for $2 \leq i \leq n-2k$. Initially, the vertices in $S_0$ can force the vertices in $S_1$ because $u_i$ forces $v_i$ for $i = 1, \ldots, 2k-2$. Then $u_{n-i+2}$ forces $u_{n-i+1}$ and $v_{k-i+1}$ forces $v_{n-i+1}$, for $2 \leq i \leq n+1$. Since all vertices are eventually forced, $Z(G) \leq |S_0| = 2k + 2$. \qed
Since some but not all of our arithmetic is modular, we define notation for the residue mod \( n \) of an integer \( \ell \) by
\[
(\ell)_n = \ell - \left\lfloor \frac{\ell}{n} \right\rfloor n.
\]

In this section we index the rows and columns of matrices and vectors starting with zero. To form the adjacency matrix of a GP graph, we order the vertices as \( u_0, u_1, u_2, \ldots, u_i, \ldots, u_{n-1} \) followed by the vertices \( v_0, v_1, v_2, \ldots, v_i, \ldots, v_{n-1} \). In the GP graph \( P(n,k) \), the neighbors of an outer cycle vertex \( u_i \) are \( v_i, u_{(i-1)_n}, \) and \( u_{(i+1)_n} \) for \( 0 \leq i \leq n-1 \). The neighbors of an inner cycle(s) vertex \( v_i \) are \( u_i, v_{(i+k)_n}, \) and \( v_{(i-k)_n} \) for \( 0 \leq i \leq n-1 \). Thus, the adjacency matrix is a block matrix of the following form
\[
\begin{bmatrix}
A(C_n) & I_n \\
I_n & A'(C_n)
\end{bmatrix}
\]
where \( A(C_n) \) is the adjacency matrix of the cycle on \( n \) vertices and the matrix \( A'(C_n) \) is a matrix with 1’s on the \( k \)th and \( (n-k) \)th super and subdiagonals and zeros elsewhere (i.e., \( A'(C_n) \) is the adjacency matrix of the inner cycle(s)).

Next we establish a technical lemma about eigenvalue multiplicities of GP graphs and then use it to show certain subfamilies of GP graphs have maximum nullity equal to zero-forcing number.

**Lemma 2.3.** Let \( n \geq 3, 0 < k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, n' = nr \) for \( r \geq 1 \), and \( k' \) satisfies \( (k')_n = k \) and \( 0 < k' \leq \left\lfloor \frac{n'-1}{2} \right\rfloor \). Suppose \( \lambda \) is an eigenvalue of \( A(P(n,k)) \) with multiplicity \( m \). Then, \( \lambda \) is an eigenvalue of \( A(P(n',k')) \) with multiplicity at least \( m \) for \( r \geq 1 \).

**Proof.** For each eigenvector of \( A(P(n,k)) \) for \( \lambda \), we construct an eigenvector of \( A(P(n',k')) \) for \( \lambda \). Since by construction independent eigenvectors of \( A(P(n,k)) \) yield independent eigenvectors of \( A(P(n',k')) \), this shows \( \lambda \) is an eigenvalue of multiplicity at least \( m \) for \( A(P(n',k')) \).

Suppose \( x = [x_j] \) is an eigenvector of \( A(P(n,k)) \) for eigenvalue \( \lambda \). By considering row \( j \) for \( 0 \leq j \leq n-1 \) and using that \( j = (j)_n \) we have
\[
x_{(j-1)_n} + x_{(j+1)_n} + x_{n+(j)_n} = (A(P(n,k))x)_j = (\lambda x)_j = \lambda x_{(j)_n}.
\]
Considering row \( j \) for \( n \leq j \leq 2n-1 \) and using that \( j-n = (j)_n \) and \( (j-n)_n = (j)_n \) yields
\[
x_{(j)_n} + x_{n+(j-k)_n} + x_{n+(j+k)_n} = (A(P(n,k))x)_j = (\lambda x)_j = \lambda x_{n+(j)_n}.
\]

We define
\[
\hat{x} = [x_0, x_1, \ldots, x_{n-1}, \ldots, x_0, x_1, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots, x_{2n-1}]^T.
\]

Observe that independence of a set of \( n \)-vectors \( x, y, \ldots \) guarantees the independence of the set of \( n' \)-vectors \( \hat{x}, \hat{y}, \ldots \) just constructed.

We show that \( \hat{x} \) is an eigenvector of \( A(P(n',k')) \) for eigenvalue \( \lambda \). Note that \( \hat{x}_j = x_{(j)_n} \) for \( j = 0, \ldots, n'-1 \), and \( \hat{x}_j = x_{n+(j-n')_n} = x_{n+(j)_n} \) for \( j = n', \ldots, 2n'-1 \). Observe that
((c)_{n'})_n = (c)_n for all integers c. Thus for \( j = 0, \ldots, n' - 1 \),

\[
(A(P(n', k')) \hat{x})_j = \hat{x}_{(j-1)_{n'}} + \hat{x}_{(j+1)_{n'}} + \hat{x}_{n'+(j)_{n'}}
= x_{(j-1)_n} + x_{(j+1)_n} + x_{n+(j)_n}
= \lambda x_{(j)_n}
= \lambda \hat{x}_j,
\]

for all integers \( j = 0, \ldots, n' - 1 \).

For \( j = n', \ldots, 2n' - 1 \)

\[
(A(P(n', k')) \hat{x})_j = \hat{x}_{(j)_{n'}} + \hat{x}_{n'+(j-k')_{n'}} + \hat{x}_{n'+(j+k')_{n'}}
= x_{(j)_n} + x_{n+(j-k)_n} + x_{n+(j+k)_n}
= \lambda x_{n+(j)_n}
= \lambda \hat{x}_j,
\]

because \((k')_n = k\) by (2).

Thus, we conclude \( A(P(n', k')) \hat{x} = \lambda \hat{x} \).

\(\square\)

**Theorem 2.4.** Let \( n \geq 3 \) and \( 0 < k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \). Suppose there is an eigenvalue \( \lambda \) of \( A(P(n, k)) \) with multiplicity \( m \). Then \( \lambda \) is an eigenvalue of \( A(P(nr, k)) \) with multiplicity at least \( m \). If \( m = 2k+2 \), then \( \text{M}(P(nr, k)) = \text{Z}(P(nr, k)) = 2k+2 \) for all integers \( r \geq 1 \). In particular, the following Generalized Petersen graphs have maximum nullity equal to zero-forcing number for \( r \geq 1 \):

(a) \( \text{M}(P(15r, 2)) = \text{Z}(P(15r, 2)) = 6 \).

(b) \( \text{M}(P(24r, 5)) = \text{Z}(P(24r, 5)) = 12 \).

**Proof.** By Lemma 2.3, \( \lambda \) is an eigenvalue of \( A(P(nr, k)) \) with multiplicity at least \( m \). When \( m = 2k + 2 \),

\[
2k + 2 = \text{null}(A(P(nr, k)) - \lambda I) \leq \text{M}(P(nr, k)) \leq \text{Z}(P(nr, k)) \leq 2k + 2,
\]

by Theorem 2.2. For (a), \(-2\) is an eigenvalue of \( A(P(15, 2)) \) with multiplicity 6. For (b), \(0\) is an eigenvalue of \( A(P(24, 5)) \) with multiplicity 12.

\(\square\)

**Theorem 2.5.** Suppose \( n \geq 3 \) and \( \lambda \) is an eigenvalue of \( A(P(n, \left\lfloor \frac{n-1}{2} \right\rfloor)) \) with multiplicity \( m \). Then for any odd positive integer \( t \), \( \lambda \) is an eigenvalue of \( A(P(nt, \left\lfloor \frac{nt-1}{2} \right\rfloor)) \) with multiplicity at least \( m \).

**Proof.** The result follows from Lemma 2.3 when it is established that \( \left\lfloor \frac{nt-1}{2} \right\rfloor \) for odd \( t \). Let \( t = 2s + 1 \) where \( s \) is a nonnegative integer. Then

\[
\left( \left\lfloor \frac{nt-1}{2} \right\rfloor \right)_n = \left( \left\lfloor \frac{n(2s+1)-1}{2} \right\rfloor \right)_n = \left( ns + \left\lfloor \frac{n-1}{2} \right\rfloor \right)_n = \left\lfloor \frac{n-1}{2} \right\rfloor.
\]

\(\square\)
Theorem 2.6. For any $k \geq 3$, $Z(P(2k + 1, k)) \leq 6$.

Proof. Define $S_0 = \{u_0, u_1, v_1, u_k, u_{k+1}, v_{k+1}\}$

$$S_i = \begin{cases} S_{i-1} \cup \left\{ u_{k+1+\frac{i+1}{2}}, v_{k+1+\frac{i+1}{2}} \right\}, & \text{for } i \text{ odd} \\ S_{i-1} \cup \left\{ u_{\frac{i+1}{2}}, v_{\frac{i+1}{2}} \right\}, & \text{for } i \text{ even} \end{cases}$$

for $1 \leq i \leq 2k - 3$, and $S_{2k-2} = S_{2k-3} \cup \{v_0, v_k\}$. We force in order of increasing $i$. For $i$ odd, $u_{k+1+\frac{i-1}{2}} \rightarrow u_{k+1+\frac{i+1}{2}}$ and $v_{\frac{i-1}{2}} \rightarrow v_{\frac{i+1}{2}}$. For $i$ even, $u_{\frac{i}{2}} \rightarrow u_{\frac{i+1}{2}}$ and $v_{k+1+\frac{i}{2}} \rightarrow v_{1+\frac{i}{2}}$. Thus $Z(P(2k + 1, k)) \leq |S_0| = 6$.

Corollary 2.7. For all odd $t \geq 1$, $M(P(15t, \frac{15t-1}{2})) = 6 = Z(P(15t, \frac{15t-1}{2}))$.

Proof. By Theorem 2.6, $Z(P(15t, \frac{15t-1}{2})) \leq 6$. It is straightforward to verify that $-2$ is an eigenvalue of $A(P(15, t))$ with multiplicity 6. So $-2$ is an eigenvalue of $A(P(15t, \frac{15t-1}{2}))$ with multiplicity at least 6 by Theorem 2.5. Thus

$$6 \leq M(P(15t, \frac{15t-1}{2})) \leq Z(P(15t, \frac{15t-1}{2})) \leq 6.$$ 


Question 2.8. Does there exist a Generalized Peterson graph $P(n, k)$ having $M(P(n, k)) < Z(P(n, k))$?

3 Expansion Procedures

In this section, we introduce expansion procedures that determine the maximum nullity and minimum rank of graphs with special characteristics. In a graph $G$, vertices $v$ and $w$ that have the same set of neighbors (except possibly $v$ and $w$) are called twins. Let $v$ and $w$ be twins; if $v$ and $w$ are not adjacent, then $v$ and $w$ are independent twins, whereas if $v$ and $w$ are adjacent, then $v$ and $w$ are adjacent twins. Define twinning of $v \in V(G)$ as a graph operation in which we add a new vertex $x$ such that $N(v) = N(x)$ (so $x$ is an independent twin of $v$). Define twin$(G, v, k)$ to be the graph resulting from performing the twinning operation $k$ times on vertex $v$ in the graph $G$.

Proposition 3.1. [7] Let $G$ be a graph with independent twin vertices $v$ and $w$. If there exists $A \in S(G - w)$ such that $\text{null}(A) = M(G)$ and the diagonal entry $a_{vv} = 0$, then $\text{mr}(G) = \text{mr}(G - w)$. Furthermore, there exists $A' \in S(G)$ with $\text{null}(A') = M(G)$ and $a'_{vv} = 0$.

Corollary 3.2. (Twinning) Suppose $M(G) = Z(G)$, $v \in V(G)$, and there exists $A = [a_{ij}] \in S(G)$ such that $\text{null}(A) = M(G)$ and $a_{vv} = 0$. Then $M(\text{twin}(G, v, k)) = Z(\text{twin}(G, v, k)) = M(G) + k$ and there exists $A' \in S(\text{twin}(G, v, k))$ with $a'_{vv} = 0$. 


Proof. By Proposition 3.1 we can conclude that $M(\text{twin}(G, v, k)) \geq M(G) + k$ and the new matrix $A'$ has $d'_{vv} = 0$. Since we can simply color an independent twin blue and add it to a given zero-forcing set we know that $Z(\text{twin}(G, v, k)) \leq Z(G) + k$. Thus, $M(G) + k \leq M(\text{twin}(G, v, k)) \leq Z(\text{twin}(G, v, k)) \leq Z(G) + k = M(G) + k$. \hfill \Box

**Theorem 3.3.** [4] (Cut vertex reduction) Let $G$ be a graph with a cut vertex $v$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Then $\text{mr}(G) = \min \left\{ \sum_{i=1}^{k} \text{mr}(G_i), \sum_{i=1}^{k} \text{mr}(G_i - v) + 2 \right\}$.

**Corollary 3.4.** Suppose that $G = G_1 \oplus G_2$ and $\text{mr}(G_i - v) = \text{mr}(G_i)$ for $i \in \{1, 2\}$. Then $\text{mr}(G) = \text{mr}(G_1) + \text{mr}(G_2)$ and $M(G) = M(G_1) + M(G_2) - 1$.

Let $S$ be a zero-forcing set of a graph $G$. A chronological list of forces is a list of the forcing operations used to color every vertex of $G$ in the order in which they are performed. A forcing chain for such a list is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that for $i = 1, 2, \ldots, k - 1$, $v_i$ forces $v_{i+1}$. A forcing chain is called maximal if it is not a proper subsequence of any other forcing chain. A reversal of $S$ is the set of last vertices of the maximal zero-forcing chains of a chronological list of forces.

**Theorem 3.5.** [3] If $S$ is a zero-forcing set of $G$ then so is any reversal of $S$.

**Lemma 3.6.** [9] Let $G$ be a graph with a cut vertex $v$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Then $Z(G) \geq \sum_{i=1}^{k} Z(G_i) - k + 1$.

**Corollary 3.7.** Suppose $G = G_1 \oplus G_2$ and for $i \in \{1, 2\}$ there exist minimum zero-forcing sets $S_i$ for $G_i$ such that $v \in S_i$. Then $Z(G) = Z(G_1) + Z(G_2) - 1$.

Proof. By Lemma 3.6 with $k = 2$ we have $Z(G) \geq Z(G_1) + Z(G_2) - 1$. To see that $Z(G) \leq Z(G_1) + Z(G_2) - 1$, note that $v$ is non-forcing in $G_1$ with respect to a reversal of $S_1$. In $G$, color blue the vertices of the reversal (|$S_1$| of them) and proceed to force the rest of the vertices of $G_1$ in $G$ by the chronological list of forces of the reversal. This leaves all of $G_1$’s vertices in $G$ colored blue, including $v$, and now by coloring the |$S_2$| - 1 other vertices of $S_2$ in $G$ we may force the rest of the graph. \hfill \Box

**Corollary 3.8.** (Cut vertex expansion) Suppose $G = G_1 \oplus G_2$. If $M(G_i) = Z(G_i)$, $\text{mr}(G_i - v) = \text{mr}(G_i)$ and there exists a minimum zero-forcing set $S_i$ for $G_i$ such that $v \in S_i$ for $i = 1, 2$, then $M(G) = Z(G)$.

4 Families of bipartite graphs constructed by expansion

In this section we apply results from the previous section to construct families of bipartite graphs, some of which are 3-semiregular.
4.1 Vertex sums of bat graphs

Define the family of bat graphs as follows: The basic bat graph, which is denoted by $B_0$, is the graph given by a $C_4$ with a leaf appended to each of two non-adjacent vertices. See Figure 4.1. This graph has $M(B_0) = Z(B_0) = 2$ with its adjacency matrix being of minimum rank. The set consisting of one leaf and one degree two vertex is a minimum zero-forcing set. In the forcing process, the other leaf does not force. Call the degree 3 vertices $x_1$ and $x_2$, the degree 2 vertices $y_1$ and $y_2$, and the leaves $y_3$ and $y_4$. Any graph constructed by a sequence of twinning operations applied to $x_1$ or $x_2$ is a bat graph.

![Figure 4.1: The basic bat graph and two expansions by twinning (each pink vertex has a newly created twin)](image)

Theorem 4.1. For every bat graph $B$, $M(B) = Z(B)$.

Proof. For the adjacency matrix of $B_0$, $\text{rank}(A(B_0)) = 4 = \text{mr}(B_0)$ and all diagonal elements are zero, so we may apply the twinning operation described in Corollary 3.4. Note that any bat graph is a 3-semiregular bipartite graph. Once at least two independent twins have been added to the $X$ set of $V(B_0)$, the resulting graph has $|X| \geq |Y|$.

Remark 4.2. If we take $G_1$ and $G_2$ to be bat graphs and let $G = G_1 \oplus G_2$ with $v$ taken to be either $y_3$ or $y_4$ in each of the $G_i$, all hypotheses of the cut vertex expansion procedure are satisfied and as a result $M(G) = Z(G)$. In particular, $y_3$ or $y_4$ may be in a minimum zero-forcing set of any bat graph, and a reversal of that set will include whichever of the two was not in the original set. Similarly, if $G$ is a vertex sum of bat graphs as described above, additional bat graphs may be appended to the $y_3$ or $y_4$ vertices of the “bat subgraphs” not already involved in a vertex sum while preserving the property that maximum nullity equals zero-forcing number, allowing the construction of chains of bats, called bat chains, of arbitrary length. See Figure 4.2.
4.2 Jewel necklace graphs

In this section, we introduce a family of graphs called jewel necklaces and show that maximum nullity equals the zero-forcing number for these graphs. Define an \( r \)-jewel to be a \( K_{r,r} \) where one edge is deleted. Define an \( s,r \)-jewel necklace, \( J_{s,r} \), for \( s \geq 2 \) and \( r \geq 3 \) to be a cycle of jewel graphs connected appropriately by edges between vertices with \( \deg(v) = r - 1 \). We require that \( r \geq 3 \) and \( s \geq 2 \) because otherwise we would be examining a cycle \( (r = 2) \), \( s \) copies of \( P_2 \) \( (r = 1) \), or a \( K_{r,r} \) \( (s = 1) \), for which maximum nullity and zero-forcing number are known. In this process of connecting \( s \) copies of \( r \)-jewels, exactly one connecting edge is incident with each vertex of degree \( r - 1 \), so \( J_{s,r} \) is \( r \)-regular. Figure 4.3 shows the general form of \( J_{s,r} \). Note that the order of \( J_{s,r} \) is \( 2sr \) and this graph is bipartite.

Theorem 4.3. The \( s,r \)-jewel necklace \( J_{s,r} \) has \( M(J_{s,r}) = Z(J_{s,r}) = 2sr - 4s + 2 \) for \( s \geq 2 \) and \( r \geq 3 \).

Proof. First we show that \( s,r \)-jewel necklace \( J_{s,r} \) has \( Z(J_{s,r}) \leq 2sr - 4s + 2 \). We number the vertices of the \( k \)th jewel as \( (k,i), i = 0, \ldots, 2r - 1 \). A zero-forcing set for the case \( s = 2 \) consists of vertices \( \{(0,0), (0,1), \ldots, (0,r-2), (0,r), \ldots, (0,2r-3), (0,2r-1), (1,1), \ldots, (1,r-2), (1,r), \ldots, (1,2r-3)\} \) and is shown in Figure 4.4. We see that vertices \( (0,1) \) and \( (0,r) \) force vertices \( (0,2r-2) \) and \( (0,r-1) \), respectively. Once these vertices are forced, vertices \( (0,0) \) and \( (0,2r-1) \) force vertices \( (1,0) \) and \( (1,2r-1) \), respectively. Now \( (1,1) \) and \( (1,r) \)
can force the final vertices. This process can be generalized for $J_{s,r}$, with the zero-forcing set shown Figure 4.3, with one jewel having its end vertices colored and the others not.

\[ \begin{array}{cccccc}
(0,0) & (0,1) & \cdots & (0,r-1) & (1,1) & (1,r-1) \\
(0,r) & & \cdots & & (1,r) & \\
(0,2r-1) & & & & (1,2r-1) \\
\end{array} \]

Figure 4.4: The double jewel, $J_{2,r}$

Twinning on a cycle can be used to construct $J_{s,r}$ and prove that $M(J_{s,r}) = Z(J_{s,r})$. We begin with $C_{4s}$ and observe that the adjacency matrix $A(C_{4s})$ has rank$(A(C_{4s})) = 4s - 2$, so $2 = M(A(C_{4s})) = Z(C_{4s})$ (and all diagonal entries of $A(C_{4s})$ are zero). We add an (independent) twin vertex of $v_i \in V(C_{4s})$ for all $i \equiv 3 \mod 4$ and for all $i \equiv 0 \mod 4$ (see Figure 4.5). Perform this twinning of $2s$ vertices $r - 2$ times to construct $J_{s,r}$. Then $mr(J_{s,r}) = 4s - 2$ by Proposition 3.1 which implies $M(J_{s,r}) = 2sr - 4s + 2$. Thus, $M(J_{s,r}) = Z(J_{s,r}) = 2sr - 4s + 2$. 

\[ \begin{array}{cccccc}
1 & 2 & \cdots & 10 & 1 & 2 \\
9 & 8 & \cdots & 4 & 9 & 8 \\
7 & 6 & & 5 & 7 & 6 \\
\end{array} \]

Figure 4.5: $C_{4s}$ before and after twinning (pink vertices selected for twinning)
5 Other cubic bipartite graphs with $M(G) = Z(G)$

In this section we show that $M(G) = Z(G)$ for two other well-known cubic bipartite graphs. We consider these graphs because this research was motivated by looking at $3$-semiregular bipartite graphs. The Bidiakis cube, $BC$, is a 12-vertex graph consisting of a cube in which two opposite faces (which we call top and bottom) have edges drawn across them which connect the centers of opposite sides of the faces in such a way that the orientation of the edges added on top and bottom are perpendicular to each other. Note that the Bidiakis cube is also isomorphic to a $C_{12}$ with edges added between three pairs of vertices on opposite sides of the cycle. This graph is shown below in Figure 5.1.

![Bidiakis Cube Diagram]

Figure 5.1: A zero-forcing set on the Bidiakis Cube and an alternate drawing

**Proposition 5.1.** The Bidiakis cube has $M(BC) = Z(BC) = 4$.

**Proof.** Since $\text{rank}(A(BC)) = 4$, we know that $M(BC) \geq 4$. As a zero-forcing set consider $S = \{1, 2, 3, 5\}$ with the vertex labeling in Figure 5.1. Consequently, $4 \leq M(BC) \leq Z(BC) \leq 4$ and so $M(BC) = Z(BC)$.

The Tutte-Coxeter graph $TC$ is defined to have a set of 30 vertices denoted by

$$a_0, a_1, \ldots, a_9, b_0, b_1, \ldots, b_9, c_0, c_1, \ldots, c_9,$$

and edges

$$\{a_i, a_{i+1}\}, \{a_i, b_i\}, \{b_i, b_{i+5}\}, \{b_i, c_i\}, \text{ and } \{c_i, c_{i+3}\},$$

where $i$ ranges over $\{0, 1, \ldots, 9\}$ and arithmetic is taken modulo 10. This graph is shown below in Figure 5.2.

![Tutte-Coxeter Graph Diagram]

**Proposition 5.2.** The Tutte-Coxeter graph has $M(TC) = Z(TC) = 10$. 

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Proof. Note that \( \text{rank}(A(TC)) = 20 \), so \( \text{mr}(TC) \leq 20 \) and \( \text{M}(TC) \geq 10 \). The set \( S = \{a_1, a_0, b_0, b_5, c_5, c_8, b_3, c_3, b_7\} \) is a zero-forcing set for the Tutte-Coxeter graph with the vertex labeling shown in Figure 5.2. Consequently, \( 10 \leq \text{M}(TC) \leq \text{Z}(TC) \leq 10 \), implying \( \text{M}(TC) = \text{Z}(TC) = 10 \).

6 Conclusion

We establish \( \text{M}(G) = \text{Z}(G) \) for the following graphs:

- The following Generalized Petersen graphs (cubic):
  - \( P(24r, 5) \) for \( r \geq 1 \) (bipartite).
  - \( P(15r, 2) \) for \( r \geq 1 \).
  - \( P(15t, \frac{15t-1}{2}) \) for odd \( t \geq 1 \).

- Bat graphs and bat chains (bipartite 3-semiregular).

- Jewel necklaces (bipartite and regular).

- The Bidiakis Cube and the Tutte-Coxeter graphs (bipartite and cubic).

We also provide expansion techniques for constructing graphs with \( \text{M}(G) = \text{Z}(G) \) from smaller graphs with \( \text{M}(G) = \text{Z}(G) \) in Section 3.
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References


