Let $G$ be a Lie group acting on a smooth manifold $M$. That is, there exists a smooth map $\Phi : G \times M \to M; (g,m) \mapsto g \cdot m$ such that $g \cdot (h \cdot m) = (gh) \cdot m$. An action is called proper if for each compact subset $A \subseteq M$, $\Phi^{-1}(A)$ is compact. A nice fact in the theory of Lie group actions is that if an action is both proper and free (meaning that it has only trivial stabilizers), then the quotient $M/G$ is again a manifold. However, if the actions is only assumed to be proper, then the quotient need not be a manifold. For example, if we think of the Lie group $SU(2)$ as a three-dimensional real manifold, with the subgroup $\mathcal{D}$ of diagonal matrices acting via conjugation, then this action is proper (because $\mathcal{D}$ is topologically the same as the unit circle, hence compact), and not free (for example, the matrix diagonal$(-1,-1)$ fixes everything), and the result of quotienting out by the action is the unit disc: no longer a smooth manifold! Instead, what is obtained is a stratified space. Loosely speaking, a stratified space is very similar to a manifold-with-boundary, except that the boundary may have non-empty boundary as well (and turtles on down), and boundaries need not be of codimension one, as happens in manifolds-with-boundary. We will give a more precise definition a little later, but note that in the example given, this is exactly the case (the unit disc has a boundary of codimension 1). Furthermore, the stratified structure of a quotient by a proper action gives information on the geometry of the original manifold. For example, if the action is both proper and free, then $M$ is a left principal fibre bundle with structure group $G$ (over the quotient manifold $M/G$). It turns out that in general, if you quotient out a manifold by a proper Lie group action, then the strata of the quotient $M/G$ correspond to “orbits of the family of all vector fields on $M/G$.”

First, let us define the notion of a differential space. A differential space $(S,C^\infty S)$ is a topological space $S$ together with a family $C^\infty S$ of real-valued functions on $S$ satisfying:

1. $\{f^{-1}(I) | f \in C^\infty S$ and $I \subseteq \mathbb{R}$ is an open interval$\}$ is a subbasis for the topology on $S$.
2. If $f_1,\ldots,f_n \in C^\infty S$ and $F \in C^\infty \mathbb{R}^n$ then $F(f_1,\ldots,f_n) \in C^\infty S$.
3. If $f : S \to \mathbb{R}$ is a function such that for each $x \in S$, there exists an open neighborhood $U$ of $x$ and a function $f_x \in C^\infty S$ satisfying $f_x|_U = f|_U$ then $f \in C^\infty S$.

For example, any manifold $M$ is a differential space $(M,C^\infty M)$ where $C^\infty M$ denotes the usual collection of smooth functions on $M$. A differential space is called subcartesian if it is Hausdorff and every $x \in S$ has a neighborhood $U$
diffeomorphic to a subset $V$ of $\mathbb{R}^n$. Note that $V$ may not be open.

One of the reasons that manifolds are such nice structures is that they come equipped with a tangent bundle, allowing one to do many linear-algebra type arguments. The corresponding notion for a subcartesian space $(S, C^\infty S)$ (or differential space in general) is that of the set of derivations of $C^\infty S$. A derivation at a point $x \in S$ is a linear map $X_x : C^\infty S \to \mathbb{R}$ such that for every $f, h \in C^\infty S$, $X_x(fh) = X_x(f)h(x) + f(x)X_x(h)$. A derivation of $C^\infty S$ is then a collection $X$ of derivations, one at each point of $S$ (so $X : S \times C^\infty S \to \mathbb{R}; (x, f) \mapsto X_x(f)$).

We denote the set of all derivations on $S$ as Der$(C^\infty S)$. An integral curve of a derivation $X \in C^\infty S$ is a smooth map $c : I \to S$ from a non-empty interval $I \subseteq \mathbb{R}$ if $\frac{d}{dt} f(c(t)) = X_{c(t)}(f)$ for every $f \in C^\infty S$ and $t \in I$. For each $x \in S$ and $X \in$ Der$(C^\infty S)$ there exists a unique maximal integral curve $c$ of $X$ such that $0 \in I$ and $c(0) = x$. For $c$ the maximal integral curve of $X$ at $x$, we denote $c(t)$ by $\exp(tX)(x)$. A vector field on a subcartesian space $(S, C^\infty S)$ is an element $X \in$ Der$(C^\infty S)$ such that there exist an open neighborhood $U$ of $x$ and $\varepsilon > 0$ such that for every $t \in (-\varepsilon, \varepsilon)$, $\exp(tX)$ is defined on $U$ and $\exp(tX)|_U$ is a diffeomorphism from $U$ onto an open subset of $S$. An orbit of a family $\mathfrak{F}$ of vector fields on a subcartesian space $S$ is defined as follows: Let $X_1, ..., X_n \in \mathfrak{F}$ and $x_0 \in S$. Define a piecewise smooth curve given by first following the integral curve of $X_1$ through $x_0$ for time $t_1$, then following the integral curve of $X_2$ through $x_1 = \exp(t_1X_1)(x_0)$ for time $t_2$, then $X_3$ through $x_2 = \exp(t_2X_2)(x_1)$ for a time $t_3$, and so on. For $j = 1, ..., n$, let $I_j$ be the closed interval in $\mathbb{R}$ with endpoints 0 and $t_j$. Then the orbit of $\mathfrak{F}$ through $x_0$ is defined as:

$$O_{x_0} = \bigcup_{n=1}^{\infty} \bigcup_{X_1, ..., X_n \in \mathfrak{F}} \bigcup_{I_1, ..., I_n} \{\exp(t_jX_j)(x_{j-1}) \in S | t_j \in I_j\}$$

where $x_j = \exp(t_jX_j)(x_{j-1})$. It will turn out that the stratified structure on $M/G$ gives information on the orbits of vector fields on $M/G$.

A stratified space $(S, \mathcal{L})$ is a second-countable subcartesian differential space $(S, C^\infty S)$ with a collection $\mathcal{L}$ of smooth manifolds with $\bigcup_{M \in \mathcal{L}} M = S$ subject to:

1. $\mathcal{L}$ is locally closed: for each $M \in \mathcal{L}$ and $x \in M$, there exists a neighborhood $U$ of $x$ in $S$ such that $M \cap U$ is closed in $U$.
2. $\mathcal{L}$ is locally finite: for each $x \in S$ there exists a neighborhood $U$ of $x$ in $S$ such that $U$ intersects only a finite number of manifolds $M \in \mathcal{L}$.
3. The Frontier Condition: For $M, N \in \mathcal{L}$, if $M \cap N \neq \emptyset$, then either $M = N$ or $M \subseteq \overline{N} \setminus N$, where $\overline{N}$ denotes the closure of $N$ in $S$.

Note that $C^\infty S \subseteq \bigcup_{M \in \mathcal{L}} C^\infty M$, but the reverse inclusion need not hold (example: manifolds with boundary).

The basic story is this: given a proper Lie group action $\Phi : G \times M \to M$, $M/G$ is a stratified space with orbit – type stratification. The orbit-type stratification is constructed as follows: Let $H$ be a compact subgroup of $G$, and define $M_{(H)} := \{x \in M | G_x = gHg^{-1} \text{ for some } g \in G\}$, where $G_x = \{g \in G | gx = x\}$ is the isotropy group of the point $x \in M$. $M_{(H)}$ is called the subset of $M$ of orbit type $H$. Let $\mathfrak{M}$ be the family of connected components of $M_{(H)}$ as $H$ varies over all
compact subgroups of $G$. $\mathcal{M}$ then yields a stratification of $M$ (not a trivial fact - actually takes quite a bit of work). To obtain a stratification $\mathcal{L}$ on $M/G$, we project the stratification $\mathcal{M}$ from $M$ to $M/G$.

Sketch of proof that $M$ is a stratified space with the stratification

$$\mathcal{M} = \{\text{connected components of } M_{(H)} | H \text{ a compact subgroup of } G\}$$

The basic idea behind the proof is to first show that for a given compact subgroup $H$ of $G$, $M_{(H)}$ is a local submanifold of $M$ (meaning that each connected component of $M_{(H)}$ is a submanifold of $M$). This will be enough to show that $\mathcal{M}$ actually is a set of manifolds covering $M$, and that it is locally closed (submanifolds of a manifold are locally closed). The way to show this is to first decompose the Lie algebra of $G$ as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie subalgebra corresponding to $H$ in $G$ and $\mathfrak{m}$ is an orthogonal subspace to it. Using this decomposition, one then shows that there is a diffeomorphism $\varphi$ from an open neighborhood $(W, S_p)$ in $\mathfrak{m} \times S_p$ onto an open neighborhood $U$ of $p \in \{x \in M|G_x = H\}$, where $S_p$ is a slice of the action at $p$. Then one shows that the set $S^H_p$ of $H$-invariant points of $S_p$ is a local submanifold of $S_p$, so $W \times S^H_p$ is a local submanifold of $W \times S_p$ and $\varphi^{-1}(W \times S^H_p) = U \cap M_{(H)}$.

Then, to show that $\mathcal{M}$ is locally finite, one proceeds by induction: if $\dim M = 0$, then $M$ is discrete, since $G$ is assumed to be connected and the action (at least) continuous, every point of $M$ is a fixed point of $G$, and there is only one orbit type: namely, $M_{(G)} = M$, so $\mathcal{M}$ is locally finite. Then assuming that we know $\mathcal{M}$ is locally finite for every $\dim N < m$ for some fixed $m \geq 1$, around each $p \in M$, we construct a $G_p$-invariant ball in $T^*S_p$, and this has dimension strictly less than $m$. Then one shows that by projecting down to $M$, using the local finiteness of the inductive assumption, and gluing the balls together, each point in $M$ is only contained in finitely many components of $M_{(H)}$ for finitely many compact subgroups $H$ of $G$.

Finally, to show that $\mathcal{M}$ satisfies the frontier condition requires quite a bit of technical details in looking at a horizontal distribution related to the action of $G$, which arises in the construction of $S_p$ using a $G$-invariant Riemannian metric. See [2], section 4.2.

It follows (after a little work showing that the quotient of these orbit-type strata is still a manifold) that $M/G$ is a stratified space with stratification $\pi(\mathcal{M})$.

**Example** [1]: In the case of $\mathcal{D}$ acting on $SU(2)$ as above, $SU(2)_{(H)} \neq \emptyset$ if and only if $H = \mathcal{D}$ or $H = \{I, -I\}$. This yields the stratification $\mathcal{M} = \{\text{diagonal matrices, matrices with non-zero off-diagonal elements}\}$. Taking the quotient

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1. We know that it covers $M$ – follows directly from the definition of a proper action that $G_p$ is compact for each $p \in M$.
2. Meaning: $S_p$ is a submanifold of $M$ containing $p$ such that:
   1. $T_pM = T_pS_p \oplus T_p(Gp)$, where $Gp = \{g \cdot p | g \in G\}$.
   2. For every $q \in S_p$, $T_qM = T_qS_p + T_q(Gq)$.
   3. $S_p$ is $G_p$ invariant.
   4. For $q \in S_p$ and $g \in G$, if $g \cdot q \in S_p$, then $g \in G_p$.

Such a slice can be constructed from a $G$-invariant Riemannian metric (deliberately vague).
SU(2)/D then yields the stratification:

\[ S_2 := \{ [A]_{\alpha,b} : |\alpha|^2 + b = 1, b \in (0, 1] \}, \]
\[ S_1 := \{ [A]_{\alpha,0} : |\alpha|^2 = 1 \} \]

where \([A]_{\alpha,b}\) is a parametrization of the equivalence classes in SU(2)/D with \(\alpha \in D(0, 1) \subset \mathbb{C}\) and \(b \in [0, 1]\), where the off-diagonal elements have norm \(b\), and the diagonal is \(\text{diagonal}(\alpha, \bar{\alpha})\). Note that as real manifolds \(S_2\) has dimension 2, \(S_1\) has dimension 1, and \(S_1\) is the boundary of \(S_2\). In fact, it is easy to see that you can smoothly map \(S_2\) to the interior of the unit disc and \(S_1\) to its boundary. (Just map \(\alpha\)!) 

**Theorem:** Strata of the orbit type stratification of \(M/G\) are orbits of the family of all vector fields on \(M/G\).

**References**


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