

OPTIMAL ERROR ESTIMATES OF THE DIRECT DISCONTINUOUS GALERKIN METHOD FOR CONVECTION-DIFFUSION EQUATIONS

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ABSTRACT. In this paper, we present the optimal L^2 -error estimate of $O(h^{k+1})$ for polynomial elements of degree k of the semidiscrete direct discontinuous Galerkin method for convection-diffusion equations. The main technical difficulty lies in the control of the inter-element jump terms which arise because of the convection and the discontinuous nature of numerical solutions. The main idea is to use some global projections satisfying interface conditions dictated by the choice of numerical fluxes so that trouble terms at the cell interfaces are eliminated or controlled. In multi-dimensional case, the orders of $k + 1$ hinge on a superconvergence estimate when tensor product polynomials of degree k are used on Cartesian grids. A collection of projection errors in both one- and multi-dimensional cases is established.

1. INTRODUCTION

In this article, we introduce an approach for proving optimal L^2 -error estimates for the semidiscrete direct discontinuous Galerkin (DDG) method solving convection-diffusion problems. To demonstrate the main idea, we will focus on the one-dimensional model equations

$$(1) \quad \partial_t U + \partial_x f(U) = \partial_x^2 U,$$

with zero flux $f = 0$, linear flux $f = \alpha U$ and nonlinear smooth flux $f(U)$, followed by multi-dimensional extensions. The DDG method was introduced in [27], refined with interface corrections in [28], and has since been extended to multi-dimensional settings as well as equations with nonlinear diffusion, for which extensive numerical tests have shown the optimal $(k + 1)$ th order of accuracy for polynomial elements of degree k . However, the optimal L^2 -error estimate has not been available. In this work we present a novel approach to obtain such an estimate. We will give the details of the proof for linear diffusion with different convections to illustrate the main ideas.

The discontinuous Galerkin (DG) method we discuss in this paper is a class of finite element methods, using a completely discontinuous piecewise polynomial space for the numerical solution and the test functions. One main advantage of the DG method was the flexibility afforded by local approximation spaces combined with the suitable design of numerical fluxes crossing cell interfaces. It was first designed and has been quite successful for solving first order PDEs such as hyperbolic

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conservation laws [15, 17, 18, 21, 31]. However, the application of the DG method to diffusion problems has been a challenging task because of the subtle difficulty in defining appropriate numerical fluxes for the solution gradient. There have been several DG methods suggested in the literature to solve the problem, including the method originally proposed by Bassi and Rebay [5] for compressible Navier-Stokes equations, its generalization called the local discontinuous Galerkin (LDG) methods introduced in [19] by Cockburn and Shu and further studied in [8, 10, 14, 16], as well as the method introduced by Baumann-Oden [6, 29]. See also the earlier works [1, 3, 35] using the so-called *interior penalty* (IP) method, and the unified analysis of DG methods in [2] for elliptic problems and background references for the IP methods.

The idea of DDG methods for higher order partial differential equations, such as the convection diffusion equation (1), is to directly force the weak solution formulation of the PDE into the DG function space for both the numerical solution and test functions. Unlike the traditional LDG method, the DDG method does not introduce any auxiliary variables or rewrite the original equation into a larger first order system. A key ingredient for the success of such methods is the correct interface corrections. These corrections must be selected to guarantee stability and solvability of the unknown to approximate the solution. The main novelty in the DDG schemes proposed in [27, 28] lies in numerical flux choices for the solution gradient, which involves high order derivatives evaluated on cell interfaces, motivated by a trace formula for the solution derivatives of the heat equation [27]. With this choice, the obtained schemes are provably stable and optimally convergent. There are other recent works also featuring the direct DG discretization, such as those by van Leer and Nomura in [25], Gassner et al. in [22], and Cheng and Shu in [20]. Stability and convergence of the schemes are presented in [20] which takes advantage of some carefully designed numerical fluxes.

Obtaining *a priori* error estimates for various DG methods has been a main subject of research. For smooth solutions of scalar conservation laws, error estimates have been given in several earlier works [24, 26, 30, 32] and the more recent one [40] for the fully discrete Runge-Kutta DG methods, and [41] for symmetrizable systems. The first *a priori* error estimate of order $O(h^k)$ for the LDG method of linear convection-diffusion was obtained in [19]. With a particular numerical flux, the optimal convergence rate of order $O(h^{k+1})$ was obtained in [7, 10, 11]. For the numerical method of Baumann and Oden [6] when applied to nonlinear convection-diffusion equations, the optimal error estimate for at least quadratic polynomials was obtained by Riviere and Wheeler [33]. The L^2 a priori error estimates for nonlinear PDEs with high order derivatives such as the KdV equations have been obtained [20, 36, 38] using certain special local projections. The optimal L^2 -error estimate for the linearized KdV equation was obtained in [37], where authors take advantage of stability estimates for auxiliary variables. A conservative discontinuous Galerkin-method for the generalized KdV equation was recently proposed by Bona et al. [4], in which a global projection was used in obtaining error estimates in some cases.

For the DDG method, the first *a priori* error estimate of order $O(h^k)$ for linear diffusion was obtained in [28]. The accurate recovery algorithm of the normal derivatives presented in [23] provides a set of effective choices of parameters in the DDG numerical fluxes. An accuracy analysis using Fourier modes for some special

solutions was presented in [39]. The main objective of this article is to present an approach to obtain the optimal *a priori* error estimate of order $O(h^{k+1})$ for polynomial elements of degree k .

In this paper, the main procedure to obtain the *a priori* error estimates is the following. First, we obtain the error equation for the DDG method. Second, we introduce a new global projection on PU : $\int_{I_j} (U - PU)v dx = 0$ for any $v \in P^{k-2}$ subject to interface conditions dictated by the DDG numerical fluxes, and prove the existence of such a projection and obtain the projection error. Third, we split the error into two parts by using the global projections: $u - PU$ and $PU - U$, which enables us to control both cell integrals and the inter-element jump terms simultaneously. In the multi-dimensional case with nonlinear convection, we use tensor product polynomials of degree at most k and show that the optimal error estimate of order $k + 1$ for $k \geq \frac{d-1}{2}$; this restriction is unnecessary for the linear convection. In multi-dimensional case, a superconvergence result is established by taking advantage of the Cartesian structure of the grid and a similar argument to that in [9] for the LDG method to solve elliptic problems on Cartesian grids.

The paper is organized as follows. In §2 we illustrate a natural way to derive the global version of the DG formulation from the PDE weak solution formulation. We further discuss how a careful choice of numerical fluxes can be made to ensure some desired features including consistency, conservation, stability and accuracy. In §3, we quantify the admissible set of numerical fluxes for the solution gradient and the convection to ensure the L^2 stability. In §4 we estimate the L^2 -error of the numerical solution from the original solution for the case of purely linear diffusion and linear convection-diffusion with two different global projections. In §5 we present the L^2 -error estimate for linear diffusion with nonlinear convection. The extension to multi-dimensions is given in §6. A collection of projection errors in both one and multi-dimensional cases is presented in §7.

2. SCHEME FORMULATION

We begin with the one-dimensional convection diffusion equation

$$(2) \quad \partial_t U + \partial_x f(U) - \partial_x^2 U = 0,$$

subject to initial data

$$(3) \quad U(x, 0) = U_0(x)$$

posed on $\Omega := [0, 1]$ with periodic boundary conditions. The weak solution formulation for this problem is to find a function $U \in C(0, T; H^1(\Omega))$ such that for all $v \in H_0^1(\Omega)$,

$$(4) \quad \langle \partial_t U, v \rangle - \langle f(U), \partial_x v \rangle + \langle \partial_x U, \partial_x v \rangle = 0, \forall v \in H_0^1(\Omega),$$

$$(5) \quad \langle U(x, 0), v \rangle = \langle U_0, v \rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product of two functions over Ω . To discretize this weak formulation, we set up a partition of the domain $\Omega = \bigcup_{j=1}^N I_j$, with mesh $I_j = [x_{j-1/2}, x_{j+1/2}]$ and mesh size $h := \Delta x = x_{j+1/2} - x_{j-1/2}$, and define the finite element space

$$\mathbb{V}_h^k := \{v \in L^2(\Omega) : v|_{I_j} \in P^k(I_j), \quad j = 1, \dots, N\},$$

where $P^k(I_j)$ denotes the space of polynomials on I_j with degree at most k . We will also adopt the following notation:

$$u^\pm = u(x \pm 0, t), \quad [u] = u^+ - u^-, \quad \{u\} = \frac{u^+ + u^-}{2}.$$

The idea of the direct discontinuous Galerkin (DDG) method in [28] is to enforce the weak formulation (4), (5) in such a way that both u and v are approximated in \mathbb{V}_h^k . The discontinuous nature of numerical solutions and test functions crossing interfaces necessarily requires some interface corrections, leading to the following:

$$(6) \quad \sum_{j=1}^N \int_{I_j} \partial_t u v dx + \sum_{j=1}^N \left(- \int_{I_j} f(u) \partial_x v dx + \int_{I_j} \partial_x u \partial_x v dx \right) + \sum_{j=1}^N \left((-\hat{f} + \widehat{\partial_x u})[v] + \widehat{\partial_x v}[u] \right)_{j+1/2} = 0,$$

$$(7) \quad \sum_{j=1}^N \int_{I_j} u(x, 0) v dx = \sum_{j=1}^N \int_{I_j} U_0 v dx.$$

Here, periodicity is realized by using same polynomials in I_1 as in I_{N+1} , and the ‘‘hat’’ terms are determined as numerical fluxes. Crucial for the scheme stability as well as for the accuracy of the DG method is the choice of numerical fluxes. The guiding principle is that numerical fluxes are chosen in such a way that they depend only on the left and right polynomials and that

- (i) they are consistent with $-f(u) + \partial_x u$ when u is smooth;
- (ii) they are conservative in the sense that they are single valued on $x_{j+1/2}$,
- (iii) they ensure stability, and
- (iv) they enforce the high order accuracy of the method.

For convection we take the numerical flux

$$(8) \quad \hat{f} = \hat{f}(u^-, u^+),$$

which is Lipschitz continuous in its arguments, and consistent with f in the sense that $\hat{f}(u, u) = f(u)$, and satisfies

$$(9) \quad \int_{u^-}^{u^+} (\hat{f}(u^+, u^-) - f(u)) du = \left(\hat{f}(u^+, u^-) - \frac{\int_{u^-}^{u^+} f(u) du}{[u]} \right) [u] \leq 0.$$

This corresponds to the entropy flux with only quadratic entropy, and it may be called the quadratic entropy flux.

For diffusion, following [28], we take $\widehat{\partial_x v} = \{\partial_x v\}$ and

$$(10) \quad \widehat{\partial_x u} = \frac{\beta_0}{h} [u] + \{\partial_x u\} + \beta_1 h [\partial_x^2 u].$$

The algorithm is now well defined once the two parameters β_i are chosen, and a particular \hat{f} is selected.

If we define two operators as

$$(11) \quad A(u, v) = \sum_{j=1}^N \int_{I_j} \partial_x u \partial_x v \, dx + \sum_{j=1}^N (\widehat{\partial_x u}[v] + [u]\{\partial_x v\})|_{j+\frac{1}{2}},$$

$$(12) \quad F(u, v) = \sum_{j=1}^N \int_{I_j} f(u) \partial_x v \, dx + \sum_{j=1}^N \hat{f}[v]|_{j+\frac{1}{2}},$$

then the scheme may be compactly written as

$$(13) \quad \langle \partial_t u, v \rangle + A(u, v) = F(u, v), \quad \forall v \in \mathbb{V}_h^k.$$

This formulation is particularly convenient to analyze the scheme.

Restrict v to only one cell I_j and we may obtain the local version

$$(14) \quad \int_{I_j} \partial_t u v \, dx = \int_{I_j} (f(u) - \partial_x u) \partial_x v \, dx + \left[(-\hat{f} + \widehat{\partial_x u})v + (u - \hat{u})\partial_x v \right] \Big|_{\partial I_j},$$

with $\hat{u} = \{u\}$, $\widehat{\partial_x u}$ defined in (10) and \hat{f} in (8).

We end this section by some further discussions on choices of the numerical flux \hat{f} . After Osher [42], a numerical flux may be called the entropy flux if it satisfies

$$\int_{u^-}^{u^+} \eta_{uu} \left(\hat{f}(u^+, u^-) - f(u) \right) du \leq 0$$

for any convex entropy function η , for it is equivalent to the E-flux as defined by

$$(15) \quad \text{sign}(u^+ - u^-) (\hat{f}(u^+, u^-) - f(u)) \leq 0$$

for all u between u^- and u^+ . A popular subclass is the so-called monotone fluxes for which \hat{f} is nondecreasing in u^- and nonincreasing in u^+ . A well-known monotone flux is the Lax-Friedrich flux of the form

$$(16) \quad \hat{f} = \{f(u)\} - \frac{\sigma}{2}[u], \quad \sigma = \max|f'|.$$

Such a choice will dissipate the entropy, and be particularly suitable for discontinuous solutions or solutions with sharp fronts when convection dominates. To summarize the relation between the different types of fluxes considered above, we have

monotone flux (16) \Rightarrow entropy flux (15) \Rightarrow quadratic entropy flux (9).

In other words, examples of monotone and entropy fluxes are actually also the quadratic entropy flux (9). In the presence of diffusion, one may take

$$(17) \quad \hat{f} = \frac{\int_{u^-}^{u^+} f(u) du}{[u]} =: \int_{u^-}^{u^+} f(u) du,$$

which is a quadratic entropy flux.

For any piecewise smooth function $u \in L^2$, on any cell interface we define

$$(18) \quad \alpha(\hat{f}; \xi) := \begin{cases} [u]^{-1}(f(\xi) - \hat{f}), & \forall \xi \in (\min(u^-, u^+), \max(u^-, u^+)), \\ \frac{1}{2} \max|f'|, & \text{if } [u] = 0, \end{cases}$$

where $\hat{f} \equiv \hat{f}(u^-, u^+)$ is an entropy satisfying flux consistent with the given flux $f(u)$. We then find that for the quadratic entropy flux (9),

$$(19) \quad \alpha(\hat{f}, u^*) \geq 0, \quad f(u^*) = \int_{u^-}^{u^+} f(u) du,$$

and for monotone fluxes we have

$$(20) \quad \alpha(\hat{f}, \xi) \geq 0, \quad \forall \xi \in (\min(u^-, u^+), \max(u^-, u^+)).$$

Notations. Throughout the paper we use unmarked norm $\|\cdot\|_B$ or $\|\cdot\|_{0,B}$ as the L^2 -norm on domain B , $\|\cdot\|_{m,B}$ as the H^m -norm and $|\cdot|_{m,B}$ as the seminorm of H^m defined by $|u|_{m,B} = (\sum_{|\alpha|=m} \int_B |\partial_x^\alpha u|^2 dx)^{1/2}$. We use $\|\cdot\|_{\infty,B}$ to denote the L^∞ -norm. B may not be included in the norm expression if it is the whole domain or the domain is clear in the context.

3. NUMERICAL FLUX AND L^2 STABILITY

We define the discrete energy norm by

$$(21) \quad \|v\|_E^2 = \sum_{j=1}^N \int_{I_j} |\partial_x v|^2 dx + \sum_{j=1}^N \frac{\beta_0}{h} [v]^2 \Big|_{j+\frac{1}{2}}, \quad v \in \mathbb{V}_h^k,$$

and introduce a quantity

$$\Gamma(\beta_1) = \sup_{v \in P^{k-1}[-1,1]} \frac{(v(1) - 2\beta_1 \partial_\xi v(1))^2}{\int_{-1}^1 v^2(\xi) d\xi},$$

where ξ may be interpreted as $2(x - x_j)/\Delta x$ for $x \in I_j$. It is clear that Γ depends on both β_1 and the polynomial degree k . Note that, for any $v_1, v_2 \in P^{k-1}([-1, 1])$ and $v = v_1 + v_2$, we have

$$(22) \quad (v(1) - 2\beta_1 \partial_\xi v(1))^2 \leq \frac{\Gamma}{2} \int_{-1}^1 v^2(\xi) d\xi \leq \Gamma \left(\int_{-1}^1 |v_1(\xi)|^2 d\xi + \int_{-1}^1 |v_2(\xi)|^2 d\xi \right).$$

This, when applied to the numerical solution u with

$$v_1(\xi) = v_1 \left(\frac{x - x_j}{h/2} \right) = \partial_x u|_{I_j}, \quad v_2(\xi) = v_2 \left(\frac{x_{j+1} - x}{h/2} \right) = \partial_x u|_{I_{j+1}},$$

gives

$$(23) \quad (2\{\partial_x u\} + \beta_1 h[\partial_x^2 u])^2 \leq \frac{2\Gamma}{h} \left(\int_{I_j} |\partial_x u|^2 dx + \int_{I_{j+1}} |\partial_x u|^2 dx \right), \quad u \in \mathbb{V}_h^k.$$

Summing over $j = 1, \dots, N$ we obtain

$$h \sum_{j=1}^N (\{\partial_x u\} + \frac{\beta_1}{2} h[\partial_x^2 u])^2 \leq \Gamma \sum_{j=1}^N \int_{I_j} |\partial_x u|^2 dx.$$

Here we have used $u|_{I_1} = u|_{I_{N+1}}$ to incorporate the periodic boundary condition. This implies that

$$\Gamma \geq \sup_{u \in P^k(I_j)} \frac{h \sum_{j=1}^N (\{\partial_x u\} + \frac{\beta_1}{2} h[\partial_x^2 u])^2}{\sum_{j=1}^N \int_{I_j} |\partial_x u|^2 dx}.$$

The right hand side is the lower bound for β_0 obtained in [28], using the admissibility criteria. We thus conclude that there exists $\gamma \in (0, 1)$ such that

$$(24) \quad A(v, v) \geq \gamma \|v\|_E^2, \quad \forall v \in \mathbb{V}_h^k,$$

provided

$$(25) \quad \beta_0 > \Gamma(\beta_1).$$

We next present a simple evaluation of Γ .

Lemma 3.1. *For any $k \geq 1$, it holds that*

$$(26) \quad \Gamma(\beta_1) = k^2 \left(1 - \beta_1(k^2 - 1) + \frac{\beta_1^2}{3}(k^2 - 1)^2 \right).$$

Moreover, $\Gamma(\beta_1)$ achieves its minimum $\frac{k^2}{4}$ at $\beta_1^* = \frac{3}{2(k^2-1)}$.

Proof. Let $\psi(\xi)$ be a vector basis function, satisfying $\int_{-1}^1 \psi_i(\xi)\psi_l(\xi)d\xi = \delta_{il}$ for $1 \leq i, l \leq k$, and let $v(\xi) = \sum_{i=1}^k a_i \psi_i(\xi)$, then

$$\int_{-1}^1 |v|^2 d\xi = a^T \left(\int_{-1}^1 \psi \psi^T d\xi \right) a = \sum_{j=1}^k |a_j|^2 = |a|^2.$$

On the other hand, for any number γ ,

$$(v(1) + \gamma \partial_\xi v(1))^2 = a^T (\psi(1) + \gamma \partial_\xi \psi(1)) (\psi(1) + \gamma \partial_\xi \psi(1))^T a \leq |\psi(1) + \gamma \partial_\xi \psi(1)|^2 |a|^2.$$

Hence for $\gamma = -2\beta_1$ as in (22) we may take

$$(27) \quad \Gamma(\beta_1) = 2|\psi(1) + \gamma \partial_\xi \psi(1)|^2.$$

To make this bound more precise, we choose the normalized Legendre polynomial vector basis $\{\psi_i\}_{i=1}^k$. Using

$$\psi_i(1) = \sqrt{\frac{2i-1}{2}}, \quad \partial_\xi \psi_i(1) = \sqrt{\frac{2i-1}{2}} \frac{i(i-1)}{2}, \quad i = 1, \dots, k,$$

we obtain for any $\gamma \in \mathbb{R}$,

$$\begin{aligned} 2|\psi(1) + \gamma \partial_\xi \psi(1)|^2 &= 2 \sum_{i=1}^k \frac{2i-1}{2} \left| 1 + \frac{\gamma}{2} i(i-1) \right|^2 \\ &= \sum_{i=1}^k (2i-1) + \gamma \sum_{i=1}^{k-1} i(i+1)(2i+1) + \frac{\gamma^2}{4} \sum_{i=1}^{k-1} i^2(i+1)^2(2i+1) \\ &= k^2 \left(1 + \frac{\gamma}{2}(k^2-1) + \frac{\gamma^2}{12}(k^2-1)^2 \right). \end{aligned}$$

The desired result follows by taking $\gamma = -2\beta_1$. Particularly, it achieves the minimum $\frac{k^2}{4}$ at $\gamma^* = -\frac{3}{k^2-1}$. \square

Lemma 3.2. *For the quadratic entropy flux \hat{f} defined in (9), it holds that*

$$F(u, u) \leq 0.$$

Proof. Using the periodic boundary condition, we calculate

$$\begin{aligned} \sum_{j=1}^N \int_{I_j} f(u) \partial_x u dx &= - \sum_{j=1}^N (f(u^*)[u])_{j+1/2} \quad \text{with } f(u^*) = \int_{u^-}^{u^+} f(u) du. \end{aligned}$$

Hence for quadratic entropy flux (9),

$$\begin{aligned} F(u, u) &= \sum_{j=1}^N (\hat{f}(u^-, u^+)[u])_{j+1/2} - \sum_{j=1}^N (f(u^*)[u])_{j+1/2} \\ &= - \sum_{j=1}^N (\alpha(\hat{f}, u^*)[u]^2)_{j+1/2} \leq 0. \end{aligned} \quad \square$$

This lemma and (24) ensures that the DDG scheme (13) has the following provable properties.

Theorem 3.3. *Consider the semidiscrete DG (13) with (β_0, β_1) satisfying (25) and the quadratic entropy flux (9) for convection, then it satisfies the following properties:*

- (1) *Conservation of mass:* $\sum_{j=1}^N \int_{I_j} u(t, x) dx = \int_{\Omega} U_0(x) dx, \quad \forall t > 0.$
- (2) *The energy $\|u\|^2 := \sum_{j=1}^N \int_{I_j} u^2 dx$ is nonincreasing in time. More precisely, there exists $\gamma \in (0, 1)$ such that*

$$(28) \quad \frac{d}{dt} \|u\|^2 \leq -2\gamma \|u\|_E^2 \leq 0.$$

- (3) *The scheme is L^2 stable in the sense that*

$$\|u\|^2 \leq \int_{\Omega} U_0^2 dx, \quad \forall t > 0.$$

Proof. (1) Taking $v = 1$ in (13) we have $\frac{d}{dt} \sum_{j=1}^N \int_{I_j} u dx = 0$, which when combined with (7) with $v = 1$ leads to the mass conservation.

- (2) Take $v = u$ in (13) and we obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + A(u, u) = F(u, u).$$

For the quadratic entropy flux, we have

$$F(u, u) \leq 0.$$

This together with the lower bound of A ensures the energy dissipation property (28).

- (3) From (2) it follows that

$$\|u\|^2 \leq \|u(\cdot, 0)\|^2 = \sum_{j=1}^N \int_{I_j} u^2(x, 0) dx \leq \sum_{j=1}^N \int_{I_j} U_0^2 dx,$$

by using (7). □

4. ERROR ESTIMATES

The idea for obtaining the optimal error estimate is to introduce some special global projection. We illustrate this idea for diffusion first, and then discuss its extension to convection-diffusion equations.

4.1. A global projection and projection error. We define a special projection P as follows. For a given smooth function w , the projection Pw is the unique function in \mathbb{V}_h^k which satisfies

$$(29a) \quad \int_{I_j} (Pw(x) - w(x))v(x)dx = 0, \quad \forall v \in P^{k-2}(I_j), \quad j = 1, \dots, N,$$

$$(29b) \quad \partial_x \widehat{(Pw)} := \beta_0 h^{-1}[Pw] + \{\partial_x(Pw)\} + \beta_1 h[\partial_x^2(Pw)] \Big|_{j+1/2} = \partial_x w(x_{j+1/2}),$$

$$(29c) \quad \{Pw\} \Big|_{j+1/2} = w(x_{j+1/2}).$$

At $x_{N+1/2}$ we use a periodic extension to be consistent with the periodic boundary condition for problems (2) and (3). For a piecewise smooth function w with $w|_{I_j} \in H^{k+1}(I_j)$, the above definition needs to be modified so that the right of (29b) and (29c) is replaced by \widehat{w}_x and $\{w\}$, respectively.

Lemma 4.1. *For (β_0, β_1) such that $\beta_0 > \Gamma(\beta_1)$, the projection P defined in (29) exists, and*

$$(30) \quad \left(\int_0^1 (Pw(x) - w(x))^2 dx \right)^{1/2} \leq Ch^{k+1}$$

for a smooth function w , where C is a constant depending on w and its derivatives but independent of h .

Proof. We first show the existence of such a projection. Define the affine map

$$x = x(\xi) = x_j + \frac{h}{2}\xi, \quad -1 \leq \xi \leq 1,$$

which maps $[-1, 1]$ onto I_j . The Legendre polynomials $\phi_k(2(x - x_j)/h) = L_{k-1}(\xi)$ are the Legendre polynomials on $[-1, 1]$, then the projection can be expressed in terms of the Legendre polynomials

$$Pw(x)|_{I_j} = \sum_{i=1}^{k+1} \alpha_i^j \phi_i(\xi), \quad 1 \leq j \leq N.$$

From (29a) with $v = \{\phi_i\}_{i=1}^{k-1}$ it follows that for $1 \leq j \leq N$,

$$\alpha_i^j = \frac{2i-1}{2} \int_{-1}^1 w(x_j + \frac{h}{2}\xi) \phi_i(\xi) d\xi, \quad i = 1, \dots, k-1,$$

where we have used $\|\phi_i\|^2 = \frac{2}{2i-1}$. It remains to determine α_k^j and α_{k+1}^j by using the interface conditions; that is, $\{Pw\}|_{j+1/2} = w(x_{j+1/2})$ and

$$\beta^0[Pw] + 2\{\partial_\xi(Pw)\} + 4\beta_1[\partial_\xi^2(Pw)] = h\partial_x w(x_{j+1/2}),$$

which lead to

$$\sum_{i=k}^{k+1} [\phi_i(1)a_i^j + \phi_i(-1)a_i^{j+1}] = b_1^j := 2w(x_{j+1/2}) - \sum_{i=1}^{k-1} [\phi_i(1)a_i^j + a_i^{j+1}\phi_i(-1)],$$

$$\sum_{i=k}^{k+1} [g_0(i)a_i^j + g_1(i)a_i^{j+1}] = b_2^j := hw_x(x_{j+1/2}) - \sum_{i=1}^{k-1} [g_0(i)a_i^j + g_1(i)a_i^{j+1}],$$

where

$$g_0(i) = -\beta_0\phi_i(1) + \phi_i'(1) - 4\beta_1\phi_i''(1), \quad g_1(i) = \beta_0\phi_i(-1) + \phi_i'(-1) + 4\beta_1\phi_i''(-1).$$

Set two matrices

$$A = \begin{pmatrix} \phi_k(1) & \phi_{k+1}(1) \\ g_0(k) & g_0(k+1) \end{pmatrix}, \quad B = \begin{pmatrix} \phi_k(-1) & \phi_{k+1}(-1) \\ g_1(k) & g_1(k+1) \end{pmatrix},$$

then the coefficient matrix of $\{a^j\}_{j=1}^N$ with $a^j = (a_k^j, a_{k+1}^j)^\top$ is an $N \times N$ block circulant matrix with the first row $[A \ B \ 0 \ \dots \ 0]$ and the last row $[B \ 0 \ \dots \ 0 \ A]$, denoted by D . The determinant of this matrix is

$$\det D = |A|^N |I + (-1)^{N-1}(A^{-1}B)^N|,$$

which is nonzero as long as $|A \pm B| \neq 0$. Recall that, for $i \geq 1$, we have

$$\begin{aligned} \phi_i(\pm 1) &= (\pm 1)^{i-1}, \quad i = 1, 2, \dots, \\ \phi_i'(\pm 1) &= \frac{1}{2}(\pm 1)^i i(i-1), \quad i = 2, 3, \dots, \\ \phi_i''(\pm 1) &= \frac{1}{8}(\pm 1)^{i-1} (i-2)(i-1)i(i+1), \quad i = 3, 4, \dots; \end{aligned}$$

hence,

$$g_0(i) = -\beta_0 + \frac{1}{2}i(i-1) - \frac{\beta_1}{2}(i-2)(i-1)i(i+1), \quad g_1(i) = (-1)^i g_0(i).$$

By a direct calculation we see that $|A \pm B| \neq 0$ is equivalent to the requirement that

$$(31) \quad g_0(i) + (-1)^i g_1(i) \neq 0, \quad i = k, k+1;$$

that is, $g_0(i) \neq 0$ for $i = k, k+1$. Under this condition the determinant of the coefficient matrix will not vanish, thus is nonsingular. The existence of global projection (29) follows. In fact, for $\beta_0 > \Gamma(\beta_1)$ we do have $g_0(i) \neq 0$ for $i = k, k+1$. This can be verified by a straightforward calculation, leading to

$$g_0(k) < -\frac{1}{2}k(k+1), \quad g_0(k+1) < -\frac{1}{2}k(k-1)$$

for $\beta_0 > \Gamma(\beta_1)$.

The desired L^2 -error estimate follows from the refined estimates stated in Lemma 4.2 below. □

Lemma 4.2 (The projection estimates). *If w is a smooth function satisfying $w \in H^{k+1}(\Omega)$, then we have the following estimates:*

$$(32) \quad \sum_{j=1}^N \|Pw - w\|_{0,I_j}^2 \leq Ch^{2k+2} |w|_{k+1,\Omega}^2,$$

$$(33) \quad \sum_{j=1}^N |Pw - w|_{j+1/2}^2 \leq Ch^{2k+1} |w|_{k+1,\Omega}^2,$$

$$(34) \quad \sum_{j=1}^N |Pw - w|_{\infty,I_j}^2 \leq Ch^{2k} |w|_{k+1,\Omega}^2,$$

where C depends on k, β_0, β_1 but is independent of I_j and w .

These projection error estimates follow from those stated in Theorem 7.1, which is proved in §7.

4.2. Projection and inverse properties. Let $\epsilon = PU - U$ denote the projection error, then Lemma 4.2 implies the following inequality:

$$(35) \quad \|\epsilon\| + h\|\epsilon\|_{\infty} + h^{1/2}\|\epsilon\|_{\Gamma_h} \leq Ch^{k+1},$$

where Γ_h denotes the set of interface points of all cells I_j , and the constant C depends on k and $|U|_{k+1}$.

For any function $e \in \mathbb{V}_h^k$, the following inverse inequalities hold [13]:

$$(36a) \quad \|\partial_x^l e\| \leq Ch^{-l}\|e\|,$$

$$(36b) \quad \|e\|_{\Gamma_h} \leq Ch^{-1/2}\|e\|,$$

$$(36c) \quad \|e\|_{\infty} \leq Ch^{-1/2}\|e\|.$$

4.3. Error analysis. If we assume the exact solution of (2) is smooth, we can obtain optimal L^2 -error estimates.

Theorem 4.3 (Diffusion without convection $f = 0$). *The solution u to the semidiscrete DDG scheme (13) for problem (2)-(3) with $f = \alpha u$ which admits a smooth solution U satisfies the following error estimate:*

$$(37) \quad \int_0^1 (u(x,t) - U(x,t))^2 dx \leq Ch^{2(k+1)},$$

where C depends on U and its derivatives but is independent of h .

Proof. The DG scheme can be written as

$$(38) \quad \langle \partial_t u, v \rangle + A(u, v) = 0$$

for all $v \in \mathbb{V}_h^k$. The scheme consistency ensures that the exact solution U of (2) also satisfies

$$(39) \quad \langle \partial_t U, v \rangle + A(U, v) = 0$$

for all $v \in \mathbb{V}_h^k$. Subtracting (38) from (39) and using the linearity of A with respect to its first arguments, we obtain the error equation

$$(40) \quad \langle \partial_t (U - u), v \rangle + A(U - u, v) = 0$$

for all $v \in \mathbb{V}_h^k$.

We now take $v = PU - u$ in the error equation (40), and denote

$$(41) \quad e = PU - u, \quad \epsilon = PU - U,$$

to obtain

$$(42) \quad \langle \partial_t e, e \rangle + A(e, e) = \langle \partial_t \epsilon, e \rangle + A(\epsilon, e).$$

For the left hand side of (42), we use the L^2 stability estimate in Theorem 3.3 to obtain

$$(43) \quad \langle \partial_t e, e \rangle + A(e, e) \geq \frac{1}{2} \frac{d}{dt} \int_0^1 e^2 dx + \gamma \|e\|_E^2.$$

As to the right hand side of (42), we first write out all the terms in $A(\epsilon, e)$:

$$\begin{aligned} A(\epsilon, e) &= \sum_{j=1}^N \int_{I_j} \partial_x \epsilon \partial_x e dx + \sum_{j=1}^{N-1} (\widehat{\partial_x \epsilon}[e] + \{\partial_x e\}[\epsilon])_{j+1/2} \\ &= - \sum_{j=1}^N \int_{I_j} \partial_x^2 \epsilon e dx + \sum_{j=1}^{N-1} (-[\partial_x \epsilon] + \widehat{\partial_x \epsilon}[e] + \{\partial_x e\}[\epsilon])_{j+1/2} \\ &= - \sum_{j=1}^N \int_{I_j} \partial_x^2 \epsilon e dx + \sum_{j=1}^{N-1} (\widehat{\partial_x \epsilon}[e] - \{\epsilon\}[\partial_x e])_{j+1/2}. \end{aligned}$$

Noticing the properties of the projection P , we have

$$\int_{I_j} \partial_x^2 \epsilon e dx = 0$$

because $\partial_x^2 \epsilon$ are polynomials of degree at most $k-2$, and at $x_{j+1/2}$,

$$\widehat{\partial_x \epsilon} = \widehat{\partial_x (PU)} - \partial_x U = 0, \quad \{\epsilon\} = \{PU\} - U = 0.$$

Therefore $A(\epsilon, e) = 0$ and the right hand side of (42) becomes

$$(44) \quad \langle \partial_t \epsilon, e \rangle \leq \frac{1}{2} \int_0^1 (\partial_t \epsilon)^2 dx + \frac{1}{2} \int_0^1 e^2 dx.$$

Plugging (43) and (44) into the equality (42) and using the approximation result in Lemma 4.1, we obtain

$$\frac{d}{dt} \int_0^1 e^2 dx + 2\gamma \|e\|_E^2 \leq \int_0^1 e^2 dx + Ch^{2k+2}.$$

The Gronwall inequality, the fact of the initial error (initial data is obtained by a standard L^2 projection),

$$\|U_0 - u(\cdot, 0)\| \leq Ch^{k+1},$$

and the approximation result in Lemma 4.1 finally give us the desired error estimate. \square

4.4. Linear convection. The same optimal error estimate can be obtained for the convection-diffusion equation with linear convection $f(u) = \alpha u$. We modify the projection as follows. For a given smooth function w , the projection Qw is the unique function in \mathbb{V}_h^k which satisfies

$$(45a) \quad \int_{I_j} (Qw(x) - w(x))v(x)dx = 0, \quad \forall v \in P^{k-2}(I_j), \quad j = 1, \dots, N,$$

$$(45b) \quad \partial_x \widehat{(Qw)} - \hat{f}(w) = \partial_x w(x_{j+1/2}) - \alpha w(x_{j+1/2}),$$

$$(45c) \quad \{Qw\} = w(x_{j+1/2}),$$

where at $x_{N+1/2}$ periodic extension of Qw is applied. For piecewise smooth function w , a similar modification to that for (29) suffices.

Lemma 4.4. *For (β_0, β_1) such that $\beta_0 > \Gamma(\beta_1)$ and any real number α , the projection Q defined in (45) exists, and*

$$(46) \quad \left(\int_0^1 (Qw(x) - w(x))^2 dx \right)^{1/2} \leq Ch^{k+1}$$

for a smooth function w , where C is a constant depending on w and its derivatives but independent of h .

Proof. The proof is entirely similar to that in Lemma 4.1, except for a modification of g_0 and g_1 ,

$$\tilde{g}_0(i) = g_0(i) - \frac{\alpha}{2}h, \quad \tilde{g}_1(i) = (-1)^i(g_0(i) + \frac{\alpha}{2}h).$$

From (31) in the proof of Lemma 4.1 it follows that the projection Q exists, provided

$$\tilde{g}_0(i) + (-1)^i \tilde{g}_1(i) \neq 0, \quad i = k, k+1.$$

In fact, the above quantity gives

$$\tilde{g}_0(i) + (-1)^i \tilde{g}_1(i) = 2g_0(i), \quad i = k, k+1.$$

Hence the result in Lemma 4.1 remains valid for the projection Q . \square

Theorem 4.5 (Diffusion with linear convection $f = \alpha u$). *The solution u to the semidiscrete DDG scheme (13) for problem (2)-(3) with $f = \alpha u$ which admits a smooth solution U satisfies the following error estimate:*

$$(47) \quad \int_0^1 (u(x, t) - U(x, t))^2 dx \leq Ch^{2(k+1)},$$

where C depends on U and its derivatives but is independent of h .

Proof. Following the proof of Theorem 4.3, we set

$$(48) \quad e = QU - u, \quad \epsilon = QU - U,$$

to obtain

$$(49) \quad (\partial_t e, e) + A(e, e) - F(e, e) = (\partial_t \epsilon, e) + A(\epsilon, e) - F(\epsilon, e).$$

For the left hand side of (49), we use the L^2 stability estimate and $F(e, e) \leq 0$ to obtain

$$(50) \quad (\partial_t e, e) + A(e, e) - F(e, e) \geq \frac{1}{2} \frac{d}{dt} \int_0^1 e^2 dx + \gamma \|e\|_E^2.$$

As to the right hand side of (49), we first write out all the terms in $A(\epsilon, e) - F(\epsilon, e)$

$$\begin{aligned} A(\epsilon, e) - F(\epsilon, e) &= - \sum_{j=1}^N \left(\int_{I_j} e_{xx} \epsilon + f(\epsilon) e_x \right) dx \\ &\quad + \sum_{j=1}^{N-1} ((\hat{\epsilon}_x - \widehat{f'(\epsilon)})[e] - \{\epsilon\}[e_x])_{j+1/2} \\ &= - \sum_{j=1}^N \int_{I_j} f(\epsilon) e_x dx, \end{aligned}$$

by the properties of the projection Q . Therefore, the right hand side of (49) becomes

$$(51) \quad (\partial_t \epsilon, e) + A(\epsilon, e) - F(\epsilon, e) \leq \frac{1}{2} \int_0^1 \epsilon_t^2 dx + \int_0^1 e^2 dx + \frac{\alpha^2}{2\gamma} \int_0^1 \epsilon^2 dx + \frac{\gamma}{2} \|e\|_E^2.$$

Plugging (50) and (51) into the equality (49) and using the approximation results in Lemma 4.4, we obtain

$$\frac{d}{dt} \int_0^1 e^2 dx + \gamma \|e\|_E^2 \leq \int_0^1 e^2 dx + Ch^{2k+2}.$$

Again by the Gronwall inequality, the initial error and the approximation result in Lemma 4.4, we obtain the desired error estimate. □

Remark 4.1. For nonlinear convection, we can still obtain the optimal error estimates, yet the above proof relying on a modified projection is no longer applicable. In the rest of this paper we present an approach to overcome the difficulty caused by the nonlinear convection, and the argument applies to linear convection as well.

5. ERROR ESTIMATES WITH NONLINEAR CONVECTION

Theorem 5.1. *The solution u to the semidiscrete DDG scheme (13) (using a monotone flux for \hat{f}) for problem (2), (3) with a smooth solution U satisfies the following error estimate:*

$$(52) \quad \int_0^1 (u(x, t) - U(x, t))^2 dx \leq Ch^{2(k+1)},$$

where C depends on U and its derivatives but is independent of h .

Proof. Let PU be defined in (29). Following the proof of Theorem 4.3, we set

$$(53) \quad e = PU - u, \quad \epsilon = PU - U,$$

to obtain

$$(54) \quad \langle \partial_t e, e \rangle + A(e, e) = \langle \partial_t \epsilon, e \rangle + A(\epsilon, e) + H, \quad H := F(U, e) - F(u, e).$$

Using the coercivity $A(e, e) \geq \gamma \|e\|_E^2$, and $A(\epsilon, e) = 0$ due to the special projection, we have

$$(55) \quad \frac{1}{2} \frac{d}{dt} \|e\|^2 + \gamma \|e\|_E^2 \leq \frac{1}{2} \int_0^1 \epsilon_t^2 dx + \frac{1}{2} \int_0^1 e^2 dx + H.$$

To estimate H from above, we take the average of u^\pm as a reference value $\{u\}$, hence

$$(56) \quad H = \sum_{j=1}^N \int_{I_j} (f(U) - f(u))e_x dx + \sum_{j=1}^N ((f(U) - f(\{u\})))[e]_{j+\frac{1}{2}} \\ + \sum_{j=1}^N ((f(\{u\}) - \hat{f}(u))[e]_{j+\frac{1}{2}}),$$

where we have taken the periodic boundary condition into account.

First we establish the estimate of the last term in (56) as

$$H_2 \leq \frac{\gamma}{3} \sum_{j=1}^N \beta_0 \frac{[e]^2}{h} + Ch^{2k+2}.$$

In fact, using notation (18) and the fact that the exact solution is continuous with $[U] = 0$ so that $[u] = [\epsilon] - [e]$, we obtain

$$H_2 = \sum_{j=1}^N (\alpha(\hat{f}; \{u\})[u][e]_{j+\frac{1}{2}}) = - \sum_{j=1}^N (\alpha(\hat{f}; \{u\})[e]^2)_{j+\frac{1}{2}} + \sum_{j=1}^N (\alpha(\hat{f}; \{u\})[\epsilon][e]_{j+\frac{1}{2}}).$$

For the monotone flux, the first term is nonpositive; see (20). Using the Young inequality, we bound the last term further by

$$\frac{\gamma}{3} \sum_{j=1}^N \beta_0 \frac{[e]^2}{h} + \frac{3h}{4\beta_0\gamma} \sum_{j=1}^N (\alpha^2[\epsilon]^2)_{j+\frac{1}{2}}.$$

The claimed bound for H_2 follows from these estimates with the projection error inequality (30),

$$\sum_{j=1}^N |[\epsilon]_{j+\frac{1}{2}}|^2 \leq C|\epsilon|_{\Gamma}^2 \leq Ch^{2k+1}.$$

To estimate the first two sums in H , we use the following Taylor expansions

$$(57) \quad f(U) - f(u) = f'(U)(e - \epsilon) - \frac{f''}{2}(e - \epsilon)^2,$$

$$(58) \quad f(U) - f(\{u\}) = f'(U)(\{e\} - \{\epsilon\}) - \frac{\tilde{f}''}{2}(\{e\} - \{\epsilon\})^2,$$

where f'' and \tilde{f}'' are the mean values. These imply the following regrouping

$$H_1 := H - H_2 = H_{11} + H_{12} + H_{13},$$

where

$$H_{11} = \sum_{j=1}^N \int_{I_j} f'(U)e\partial_x e dx + \sum_{j=1}^N (f'(U)\{e\}[e]_{j+\frac{1}{2}}), \\ H_{12} = - \left(\sum_{j=1}^N \int_{I_j} f'(U)\epsilon\partial_x e dx + \sum_{j=1}^N (f'(U)\{\epsilon\}[e]_{j+\frac{1}{2}}) \right), \\ H_{13} = -\frac{1}{2} \left(\sum_{j=1}^N \int_{I_j} f''(e - \epsilon)^2 \partial_x e dx + \sum_{j=1}^N (\tilde{f}''(\{e\} - \{\epsilon\})^2 [e]_{j+\frac{1}{2}}) \right)$$

will be estimated separately as below.

For the H_{11} term, a simple integration by parts gives

$$H_{11} = -\frac{1}{2} \sum_{j=1}^N \int_{I_j} f''(U) \partial_x U e^2 dx \leq C \|e\|^2.$$

Using Young's inequality and (30) we obtain

$$\begin{aligned} H_{12} &\leq \frac{\gamma}{3} \sum_{j=1}^N \int_{I_j} e_x^2 dx + \frac{3}{4\gamma} \sum_{j=1}^N \int_{I_j} (f'(U)\epsilon)^2 dx \\ &\quad + \frac{\gamma}{3} \sum_{j=1}^N \beta_0 \frac{[e]^2}{h} + \frac{3h}{4\beta_0\gamma} \sum_{j=1}^N (f'(U)\{\epsilon\})_{j+\frac{1}{2}}^2 \\ &\leq \frac{\gamma}{3} \|e\|_E^2 + Ch^{2k+2}. \end{aligned}$$

To deal with the higher order terms we use both projection and inverse inequalities, (30) and (36), to get

$$\begin{aligned} H_{13} &\leq \frac{\gamma}{3} \sum_{j=1}^N \int_{I_j} (\partial_x e)^2 dx + \frac{3}{16\gamma} \sum_{j=1}^N \int_{I_j} (f''(e - \epsilon))^2 dx \\ &\quad + \frac{\gamma}{3} \sum_{j=1}^N \beta_0 \frac{[e]^2}{h} + \frac{3h}{16\beta_0\gamma} \sum_{j=1}^N (\tilde{f}''\{e - \epsilon\})_{j+\frac{1}{2}}^2 \\ &\leq \frac{\gamma}{3} \|e\|_E^2 + C \|e - \epsilon\|_\infty^2 (\|e\|^2 + \|\epsilon\|^2 + h\|\epsilon\|_{\Gamma_h}^2 \\ &\quad + h\|\epsilon\|_{\Gamma_h}^2 + h\|e\|_{\Gamma_h}^2). \end{aligned}$$

Plugging all above estimates on H into the inequality (54), we obtain

$$(59) \quad \frac{d}{dt} \|e\|^2 \leq C(\|e\|^2 + h^{2k+2})(1 + \|e - \epsilon\|_\infty^2).$$

Using the approximation results in (30) and (36), we have

$$(60) \quad \begin{aligned} \frac{d}{dt} \|e\|^2 &\leq C(\|e\|^2 + h^{2k+2})(1 + h^{-1}\|e\|^2 + h^{2k+1}) \\ &\leq C[\|e\|^2(1 + h^{-1}\|e\|^2) + h^{2k+2}]. \end{aligned}$$

Note that for u_0 to be the standard L^2 projection of U_0 the initial error gives

$$\|e_0\| = \|U_0 - u^0\| \leq Ch^{k+1}.$$

For fixed finite time interval $[0, T]$, integration of (60) over $[0, t]$ for $t \leq T$ gives

$$\|e(t)\|^2 \leq C(T)[h^{2k+2} + \int_0^t \|e(\tau)\|^2(1 + h^{-1}\|e(\tau)\|^2)d\tau] := CW.$$

This implies that $W(0) = h^{2k+2}$, and

$$(61) \quad \frac{d}{dt} W = \|e(t)\|^2(1 + h^{-1}\|e(t)\|^2) \leq CW(1 + Ch^{-1}W).$$

We define a constant \tilde{C} depending on T such that

$$CT = \int_1^{\tilde{C}} \frac{d\eta}{\eta(1 + C\eta h^{2k+1})} =: G(\tilde{C}),$$

which ensures a uniformly bound \tilde{C} in terms of h . Integration of inequality (61) leads to

$$G(W/W(0)) \leq CT = G(\tilde{C}).$$

Hence $W(T) \leq \tilde{C}W(0) = \tilde{C}h^{2k+2}$, leading to $\|e\|^2 \leq CW \leq Ch^{2k+2}$, implying the desired error estimate. \square

6. EXTENSION TO MULTI-DIMENSIONS

The extension to multi-dimensions with rectangular meshes is straightforward, for which we need to first clarify some notations. Consider the following model equation

$$(62) \quad \partial_t U + \nabla_x \cdot f(U) = \Delta_x U, \quad t > 0,$$

posed on $x = (x^1, \dots, x^d) \in \Omega = \prod_{i=1}^d [0, L_{x^i}] \subset \mathbb{R}^d$ with periodic boundary conditions. Here $f: \mathbb{R} \rightarrow \mathbb{R}^d$ is a smooth vector function.

6.1. Scheme formulation. For a rectangular partition of

$$\Omega = \bigcup_{\alpha=1}^N K_\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$, $N = (N_1, \dots, N_d)$. Here we use rectangular meshes $\{K\} \subset T_h$, with $K_\alpha = I_{\alpha_1}^1 \times \dots \times I_{\alpha_d}^d$, where $I_{\alpha_i}^i = [x_{\alpha_i-1/2}^i, x_{\alpha_i+1/2}^i]$ for $\alpha_i = 1, \dots, N_i$. The cell lengths are denoted by $h^{x^i} = \max_{1 \leq \alpha_i \leq N_i} |I_{\alpha_i}^i|$, with $h = \max_{1 \leq i \leq d} h^{x^i}$ being the maximum mesh size. We again assume that the mesh is regular.

We define the DG space as the space of tensor product of piecewise polynomials of degree at most k in each variable on every element, i.e.,

$$\mathbb{W}_h^k = \{v : v \in Q^k(K_\alpha), \forall x \in K_\alpha, \alpha = 1, \dots, N\},$$

where Q^k is the space of tensor products of one-dimensional polynomials of degree up to k . For the one-dimensional case, we have $Q^k(K) = P^k(K)$, which is the space of polynomials of degree at most k defined on K .

We also introduce some trace operators that will help us to define the interface terms. Let K^1 and K^2 be two neighboring cells with a common edge e ; for w defined on ∂K^i , $i = 1, 2$, we define the average $\{w\}$ and the jump $[w]$ as

$$\{w\} = \frac{1}{2}(w_1 + w_2), \quad [w] = w_2 - w_1 \quad \text{on } e,$$

where the jump is calculated as a forward difference along the normal direction \vec{n} , which is defined to be oriented from K^1 to K^2 , with $w_i = w|_{\partial K^i}$. We start with the weak formulation of (62) of the form

$$\int_{\Omega} \partial_t U v dx + \int_{\Omega} \nabla U \cdot \nabla v dx = \int_{\Omega} f(U) \cdot \nabla_x v dx, \quad \forall v \in H_0^1(\Omega),$$

and approximate U by $u \in \mathbb{W}_h^k$, and also allow $v \in \mathbb{W}_h^k$ to be discontinuous crossing ∂K . With such a DG discretization the above formulation has to be refined by including some interface corrections, leading to the following DG scheme,

$$(63) \quad \langle u_t, v \rangle + A(u, v) = F(u, v),$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product over Ω , and

$$(64a) \quad A(u, v) = \sum_{K \in T_h} \int_K \nabla_x u \cdot \nabla_x v \, dx + \sum_{e \in \Gamma} \int_e (\widehat{\partial_n u}[v] + [u]\{\partial_n v\}) \, ds,$$

$$(64b) \quad F(u, v) = \sum_{K \in T_h} \int_K f(u) \cdot \nabla_x v \, dx + \sum_{e \in \Gamma} \int_e \widehat{f^n}[v] \, ds.$$

This is a complete DG discretization as long as the “hat” terms are defined.

6.2. Stability. For nonlinear flux we take \hat{f} as an entropy satisfying numerical flux: let $e = \partial K^1 \cap \partial K^2$ and $u_i = u|_{\partial K^i}$,

$$(65) \quad \int_e (\widehat{f^n}(u_1, u_2) - f(\xi) \cdot \vec{n})[u] \, ds \leq 0$$

for all ξ between u_1 and u_2 . For the directional derivative of the solution induced from the diffusion we take

$$\widehat{\partial_n u} = \beta_0 h^{-1}[u] + \{\partial_n u\} + \beta_1 h[\partial_n^2 u]$$

with (β_0, β_1) chosen so that β_0 is suitably large ($> \Gamma(\beta_1)$) to ensure the following coercivity

$$(66) \quad A(v, v) \geq \gamma \|v\|_E^2$$

for some $\gamma > 0$, where

$$(67) \quad \|v\|_E^2 = \sum_{K \in T_h} \int_K |\nabla_x v|^2 \, dx + \sum_{e \in \Gamma} \int_e \frac{\beta_0}{h} [v]^2 \, ds, \quad v \in \mathbb{W}_h^k.$$

Lemma 6.1. *For the entropy flux \hat{f} defined in (65), it holds that*

$$F(v, v) \leq 0, \quad \forall v \in \mathbb{W}_h^k.$$

Proof. Set $g'_i(v) = f_i(v)$ with $g(v) = (g_1(v), \dots, g_d(v))$, then

$$\int_K f(v) \cdot \nabla_x v \, dx = \int_K \nabla_x \cdot g(v) \, dx = \int_{\partial K} g(v) \cdot \vec{n}_K \, ds,$$

which can be rewritten as

$$\int_{\partial K} \{g(v)\} \cdot \vec{n}_K \, ds - \frac{1}{2} \int_{\partial K} [g(v)] \cdot \vec{n}_K \, ds.$$

Hence, using (65) we have

$$\begin{aligned} F(v, v) &= \sum_{e \in \Gamma} \left[- \int_e [g] \cdot \vec{n} \, ds + \int_e \widehat{f^n}[v] \, ds \right] \\ &= \sum_{e \in \Gamma} \int_e (\widehat{f^n} - f(\xi) \cdot \vec{n})[v] \, ds \leq 0, \end{aligned}$$

where we have used the mean value theorem and the definition of g ; ξ is between $v|_{\partial K}$ and v^{ext} . □

Both $A(u, u) \geq \gamma \|u\|_E^2$ and $F(u, u) \leq 0$ imply the desired energy stability

$$(68) \quad \frac{d}{dt} \|u\|^2 \leq -\gamma \|u\|_E^2 \leq 0.$$

6.3. Projection and projection properties. To prove the error estimates for multi-dimensional problems with rectangular meshes, we need a suitable projection similar to the one-dimensional case. Such a projection can be defined as

$$(69) \quad \Pi w = P^{(x^1)} \otimes \cdots \otimes P^{(x^d)} w,$$

where the superscripts indicate the application of the one-dimensional operator $P^{(x^i)}$ with respect to the corresponding variable x^i .

In other words, for a given (piecewise) smooth function w , the projection Πw is the unique function in \mathbb{W}_h^k defined in (69), with $P^{(x^i)}$ determined by

$$(70a) \quad \int_{I_{\alpha_i}^i} (P^{(x^i)} w(x) - w(x)) \partial_{x^i}^2 v(x) dx^i = 0, \quad \forall v \in P^k(I_{\alpha_i}^i), \quad \alpha_i = 1, \dots, N_i,$$

$$(70b) \quad \partial_{x^i} (\widehat{P^{(x^i)} w}) := \beta_0 (h^i)^{-1} [P^{(x^i)} w] + \{\partial_{x^i} (P^{(x^i)} w)\} + \beta_1 h^i [\partial_{x^i}^2 (P^{(x^i)} w)] \Big|_{x_{\alpha_i+1/2}^i} = \widehat{\partial_{x^i} w} \Big|_{x_{\alpha_i+1/2}^i},$$

$$(70c) \quad \{P^{(x^i)} w\} \Big|_{x_{\alpha_i+1/2}^i} = \{w\} \Big|_{x_{\alpha_i+1/2}^i},$$

where periodic extensions are used at the domain boundary. Similar to the one-dimensional case, there are some approximation results for the above multi-dimensional projection:

$$(71) \quad \|\epsilon\| + h^{\frac{d+1}{2}} \|\epsilon\|_{\infty} + h^{1/2} \|\epsilon\|_{\Gamma_h} \leq Ch^{k+1},$$

where $\epsilon = \Pi U - U$. The positive constant C , solely depending on $|U|_{k+1}$, is independent of h . Γ_h denotes the boundary sets of all elements K_{α} . We present more refined estimates than (71) in Theorem 7.3. The proof presented in §7 for the two-dimensional setting is for simplicity only.

Finally, we list some inverse properties of the finite element space \mathbb{W}_h^k that will be used in our error analysis. For any function $w_h \in \mathbb{W}_h^k$, the following inverse inequalities hold [13]:

$$(72a) \quad \|\partial_x^l w_h\| \leq Ch^{-l} \|w_h\|,$$

$$(72b) \quad \|w_h\|_{\Gamma_h} \leq Ch^{-1/2} \|w_h\|,$$

$$(72c) \quad \|w_h\|_{\infty} \leq Ch^{-d/2} \|w_h\|,$$

where d is the spatial dimension. For more details of these inverse properties, we refer to [13].

6.4. Error estimates. The stability result also yields the following error estimate.

Theorem 6.2. *Let u be the solution to the semidiscrete DDG scheme (63) for problem (62) subject to initial data U_0 using polynomial elements of degree k , and U be a smooth solution of (62) subject to initial U_0 . If $k \geq \frac{d-1}{2}$, then the following error estimate holds:*

$$(73) \quad \int_{\Omega} (u(x, t) - U(x, t))^2 dx \leq Ch^{2(k+1)},$$

where C depends solely on $|U|_{k+2}$ and data given, but is independent of h .

Remark 6.1. Note that for purely diffusion problem with no convection (i.e., $f(u) = 0$) or with linear diffusion $f(u) = cu$, the restriction $k \geq \frac{d-1}{2}$ is unnecessary. For nonlinear convection, this restriction suggests that one may use polynomials of degree $k \geq 1$ for $d = 2, 3$.

Remark 6.2. In the multi-dimensional case, the proof requires a stronger smoothness assumptions on the exact solution than the ones in the one-dimensional case. In this sense, Theorem 6.2 may be regarded as a superconvergence result.

Proof. Let $\tilde{U} \in \mathbb{W}_h^k$ be determined to approximate U , and set

$$(74) \quad w = \tilde{U} - u, \quad \epsilon = \tilde{U} - U,$$

so that $U - u = w - \epsilon$. We take $v = w$ in (63) to obtain

$$(75) \quad \langle \partial_t w, w \rangle + A(w, w) = \langle \partial_t \epsilon, w \rangle + A(\epsilon, w) + H, \quad H := F(U, w) - F(u, w).$$

The choice of (β_0, β_1) can be made so that

$$(76) \quad A(w, w) \geq \gamma \|w\|_E^2.$$

We take $\tilde{U} = \Pi U$ as the projection of U defined in (70). In multi-dimensional case, $A(\epsilon, w)$ is no longer zero, but still controllable with the following superconvergence result.

Lemma 6.3. *Let A be defined in (64a). Then we have for any $s \geq 0$,*

$$(77) \quad A(\epsilon, w) \leq C \left(\|w\|^2 + h^{2 \min\{k, s\} + 2} \right) + \frac{1}{2} A(w, w).$$

The constant C depends on $|U|_{s+2}$.

We postpone the proof of this result to the end of this section, and proceed to estimate H in (75). We take the average of $u|_{\partial K^i}$ as a reference value $\{u\}$, hence

$$(78) \quad H = \sum_{K \in T_h} \int_K (f(U) - f(u)) \cdot \nabla_x w dx + \int_{\Gamma} (f^n(U) - f^n(\{u\})) [w] ds \\ + \int_{\Gamma} (f^n(\{u\}) - \hat{f}^n) [w] ds,$$

where we use the notation $f^n(\xi) = f(\xi) \cdot \vec{n}$ and have taken the periodic boundary condition into account.

Using the Taylor expansions

$$(79) \quad f(U) - f(u) = f'(U)(w - \epsilon) - \frac{f''}{2}(w - \epsilon)^2,$$

$$(80) \quad f(U) - f(\{u\}) = f'(U)(\{w\} - \{\epsilon\}) - \frac{\tilde{f}''}{2}(\{w\} - \{\epsilon\})^2,$$

where f'' and \tilde{f}'' are the mean values, and $\alpha^n(\hat{f}; \xi) := [u]^{-1}(f(\xi) - \hat{f})$ with $[u] = [\epsilon] - [w]$ we obtain

$$H = H_1 + H_2 + H_3 + H_4,$$

where

$$\begin{aligned} H_1 &= \sum_{K \in T_h} \int_K w f'(U) \cdot \nabla_x w dx + \int_{\Gamma} \vec{n} \cdot f'(U) \{w\} [w] ds, \\ H_2 &= - \left(\sum_{K \in T_h} \int_K \epsilon f'(U) \cdot \nabla_x w dx + \int_{\Gamma} \vec{n} \cdot f'(U) \{\epsilon\} [w] ds \right), \\ H_3 &= - \frac{1}{2} \left(\sum_{K \in T_h} \int_K (w - \epsilon)^2 f'' \cdot \nabla_x w dx + \int_{\Gamma} \vec{n} \cdot \tilde{f}''(\{w\} - \{\epsilon\})^2 [w] ds \right), \\ H_4 &= - \int_{\Gamma} \alpha^n(\hat{f}; \{u\}) [w]^2 ds + \int_{\Gamma} \alpha^n(\hat{f}; \{u\}) [\epsilon] [w] ds \end{aligned}$$

will be estimated separately as below.

For the H_1 term, a simple integration by parts gives

$$\begin{aligned} H_1 &= \int_{\Gamma} \vec{n} \cdot f'(U) \{w\} [w] ds - \frac{1}{2} \int_{\Gamma} \vec{n} \cdot f'(U) [w^2] ds + \frac{1}{2} \sum_{K \in T_h} \int_K f''(U) \cdot \nabla_x U w^2 dx \\ &= \frac{1}{2} \sum_{K \in T_h} \int_K f''(U) \cdot \nabla_x U w^2 dx \leq C \|w\|^2. \end{aligned}$$

Using Young's inequality and (71) we obtain

$$\begin{aligned} H_2 &\leq \frac{\gamma}{6} \sum_{K \in T_h} \int_K |\nabla_x w|^2 dx + \frac{3}{2\gamma} \sum_{K \in T_h} \int_K (f'(U) \epsilon)^2 dx \\ &\quad + \frac{\gamma}{6} \int_{\Gamma} \beta_0 \frac{[w]^2}{h} ds + \frac{3h}{2\beta_0\gamma} \int_{\Gamma} (f'(U) \epsilon)^2 ds \\ &\leq \frac{\gamma}{6} \|w\|_E^2 + Ch^{2k+2}. \end{aligned}$$

To deal with the higher order terms we use both projection and inverse inequalities, (71) and (72), to get

$$\begin{aligned} H_3 &\leq \frac{\gamma}{6} \sum_{K \in T_h} \int_K |\nabla_x w|^2 dx + \frac{3}{8\gamma} \sum_{K \in T_h} \int_K (f''(w - \epsilon))^2 dx \\ &\quad + \frac{\gamma}{6} \int_{\Gamma} \beta_0 \frac{[w]^2}{h} ds + \frac{3h}{8\beta_0\gamma} \int_{\Gamma} (\tilde{f}'' \{w - \epsilon\})^2 ds \\ &\leq \frac{\gamma}{6} \|w\|_E^2 + C \|w - \epsilon\|_{\infty}^2 (\|w\|^2 + \|\epsilon\|^2). \end{aligned}$$

For the entropy flux, the first term in H_4 is nonpositive; using the Young inequality, we bound the last term further by

$$\begin{aligned} H_4 &\leq \frac{\gamma}{6} \int_{\Gamma} \beta_0 \frac{[w]^2}{h} ds + \frac{3h}{2\beta_0\gamma} \int_{\Gamma} ((\alpha^n)^2 [\epsilon]^2) ds \\ &\leq \frac{\gamma}{6} \int_{\Gamma} \beta_0 \frac{[w]^2}{h} ds + Ch^{2k+2}. \end{aligned}$$

The above estimates lead to

$$H \leq \frac{\gamma}{2} \|w\|_E^2 + C(\|w\|^2 + h^{2k+2})(1 + \|w - \epsilon\|_{\infty}^2).$$

Substituting this, (76) and (77) into the equality (75), we obtain

$$(81) \quad \frac{d}{dt} \|w\|^2 \leq C(\|w\|^2 + h^{2k+2})(1 + \|w - \epsilon\|_\infty^2).$$

Using the approximation results (71) and (72), for small h , we have

$$\|w - \epsilon\|_\infty \leq \|w\|_\infty + \|\epsilon\|_\infty \leq C(h^{-d/2}\|w\| + h^{k+1-\frac{d+1}{2}}) \leq C(h^{-d/2}\|w\| + 1),$$

where we have used the assumption that $k \geq \frac{d-1}{2}$. Hence (81) reduces to

$$(82) \quad \frac{d}{dt} \|w\|^2 \leq C [\|w\|^2(1 + h^{-d}\|w\|^2) + h^{2k+2}].$$

Using the same argument as that estimating (60), with G replaced by

$$CT = G(\tilde{C}) = \int_1^{\tilde{C}} \frac{d\eta}{\eta(1 + C\eta h^{2k+2-d})},$$

which defines a uniform bound of \tilde{C} in terms of h since $2k + 2 - d \geq 1$, we obtain that $\|w\|^2 \leq C\tilde{C}h^{2k+2}$, leading to the desired error estimate. \square

Proof of Lemma 6.3. Let us use the notation $\int_\Gamma := \sum_{e \in \Gamma} \int_e$. A straightforward calculation shows that for all $K \in T_h$,

$$\sum_{K \in T_h} \int_{\partial K} \epsilon \partial_n w ds = - \int_\Gamma [\epsilon] \cdot \{\partial_n w\} ds - \int_\Gamma \{\epsilon\} \cdot [\partial_n w] ds,$$

with which we have

$$\begin{aligned} A(\epsilon, w) &= \sum_{K \in T_h} \int_K \nabla_x \epsilon \cdot \nabla_x w dx + \int_\Gamma (\widehat{\partial_n \epsilon}[w] + \{\epsilon\} \{\partial_n w\}) ds \\ &= - \sum_{K \in T_h} \int_K \Delta_x w \epsilon dx + \int_\Gamma (\widehat{\partial_n \epsilon}[w] - \{\epsilon\} \{\partial_n w\}) ds. \end{aligned}$$

In multi-dimensional case, this is no longer zero. To estimate it, we rewrite

$$A(\epsilon, w) = \sum_{i=1}^d A_i(U, w),$$

where

$$A_i(U, w) = - \sum_{\alpha=1}^N \int_{K_\alpha} w_{x^i x^i} \epsilon dx + \int_{K_\alpha / I_{\alpha_i}^i} \left(\widehat{\partial_{x^i} \epsilon}[w] - \{\epsilon\} \{\partial_{x^i} w\} \right)_{x_{\alpha_i}^{i+1/2}} d\hat{x}^i$$

with $\epsilon = \Pi U - U$ and $d\hat{x}^i = \prod_{j \neq i} dx^j$. The proof of the approximation results for $A_i, i = 1 \cdots d$ is analogous; therefore we only present the one for A_1 . Here we use an argument similar to the one in the proof of Lemma 3.6 in [9] for a local projection, with the unsettled part further absorbed by $A(w, w)$.

We first claim that

$$(83) \quad A_1(U, w) = 0, \quad \forall U|_{K_\alpha} \in P^{k+1}(K_\alpha), \quad w \in \mathbb{W}_h^k.$$

To prove this claim, we fix $w \in \mathbb{W}_h^k$. Since Π is a polynomial preserving operator, (83) holds true for every $U \in \mathbb{W}_h^k$. Therefore, we only need to consider the cases

$$U(x)|_{K_\alpha} = a_\alpha (x^j)^{k+1},$$

where constant a_α may vary from element to element. Below we shall use the notation $U_i = U(x^i)$ to denote the dependence only on variable x^i .

For $j = 1$, we have $\Pi U = P^{(x^1)}U_1$ and $w_{x^1x^1}$ is a polynomial of degree at most $k - 2$ in x^1 , we obtain

$$\sum \int_{K_\alpha} (\Pi U - U)w_{x^1x^1} dx = \sum \int_{K_\alpha} (P^{(x^1)}U_1 - U_1)w_{x^1x^1} dx = 0.$$

In addition, we have

$$\{\Pi U\} = \{P^{(x^1)}U_1\} = \{U\}, \quad \partial_{x^1}(\widehat{\Pi U}) = \partial_{x^1}(\widehat{P^{(x^1)}U_1}) = \partial_{x^1}\widehat{U}.$$

Thus, $A_1(U, w) = 0$ for $U|_{K_\alpha} = a_\alpha(x^1)^{k+1}$.

In the case $j \neq 1$, we integrate by parts and obtain

$$A_1(U, w) = - \sum \int_{K_\alpha/I_1^{x^1}} d\hat{x}^1 \left(- \int_{I_1^{x^1}} w_{x^1} \partial_{x^1} \epsilon dx^1 + \left(\partial_{x^1} \widehat{\epsilon}[w] + \{w_{x^1}\}[\epsilon] \right)_{x_{\alpha_1+1/2}^1} \right).$$

Due to the special form of U we have $\partial_{x^1} \epsilon = \partial_{x^1}(\Pi U - U) = 0$ and

$$\Pi U = P^{(x^j)}U_j$$

on the interface $x^1 = x_{\alpha_1+1/2}^1$, where $\partial_{x^1} \widehat{\epsilon} = 0$ and $[\epsilon] = 0$ by a direct check. We thus conclude that $A_1(U, w) = 0$ for $U|_{K_\alpha} = a_\alpha(x^j)^{k+1}$. This completes the proof of (83).

For fixed $w \in \mathbb{W}_h^k$, the linear functional $U \rightarrow A_1(U, w)$ is continuous on H^{s+2} with norm bounded by $C [\sum_\alpha \|w\|_{2, K_\alpha}^2]^{1/2}$. Due to (83), it vanishes over P^{s+1} for any $0 \leq s \leq k$. Hence, by applying the Bramble-Hilbert lemma combined with the standard scaling argument, we obtain for $U \in H^{s+2}$ that

$$|A_1(U, w)| \leq Ch^{s+2} |U|_{s+2} \left[\sum_\alpha \|w\|_{2, K_\alpha}^2 \right]^{1/2}.$$

Applying this to all A_i and using the inverse inequality (72a), we proceed with

$$\begin{aligned} |A(U, w)| &\leq Ch^{s+2} |U|_{s+2} \left(\|w\|^2 + (1 + Ch^{-2}) \sum_\alpha \|\nabla_x w\|_{0, K_\alpha}^2 \right)^{1/2} \\ &\leq Ch^{s+2} |U|_{s+2} (\|w\|^2 + h^{-2} \|w\|_E^2)^{1/2} \\ &\leq Ch^{s+2} |U|_{s+2} (\|w\| + h^{-1} \|w\|_E) \\ &\leq Ch^{s+2} |U|_{s+2} \|w\| + \frac{C^2}{2\gamma} |U|_{s+2}^2 h^{2s+2} + \frac{\gamma}{2} \|w\|_E^2 \\ &\leq C(|U|_{s+2}) (\|w\|^2 + h^{2s+2}) + \frac{1}{2} A(w, w). \end{aligned}$$

Hence we have proved Lemma 6.3. □

7. SOME ESTIMATES FOR THE PROJECTION ERROR

Theorem 7.1 (The projection estimates). *Let $P : H^{s+1} \rightarrow \mathbb{V}_h^k$ denote the one-dimensional global projection (29). If $w|_{I_j} \in H^{s+1}(I_j)$ for $j = 1, \dots, N$, then we have the following estimates:*

$$(84) \quad \sum_{j=1}^N \|Pw - w\|_{0,I_j}^2 \leq Ch^{2p+2} \sum_{j=1}^N |w|_{p+1,I_j}^2,$$

$$(85) \quad \sum_{j=1}^N |Pw - w|_{j+1/2}^2 \leq Ch^{2p+1} \sum_{j=1}^N |w|_{p+1,I_j}^2,$$

$$(86) \quad \sum_{j=1}^N |Pw - w|_{\infty,I_j}^2 \leq Ch^{2p} \sum_{j=1}^N |w|_{p+1,I_j}^2,$$

for any $1 \leq p \leq \min\{k, s\}$, where C depends on k, β_0, β_1 but is independent of I_j and w .

This leads to the estimates stated in Lemma 4.2.

We present a self-contained proof using Legendre polynomials; see [34] for details of using Legendre polynomials to estimate the local projection error.

Proof. This proof is carried out in four steps.

Step 1. First, we derive bounds on the difference $w - Pw$ in terms of the Legendre coefficients of w . To do so, denote by $\phi_{i+1} = L_i(\xi), i \geq 0$, the Legendre polynomial of degree i on $I = [-1, 1]$ and expand the function w and Pw on I_j into the series

$$(87) \quad w|_{I_j} = \tilde{w}_j(\xi) := \sum_{i=1}^{\infty} \omega_i^j \phi_i(\xi),$$

$$(88) \quad Pw|_{I_j} = \widetilde{Pw}_j(\xi) := \sum_{i=1}^{k-1} \omega_i^j \phi_i(\xi) + a_k^j \phi_k(\xi) + a_{k+1}^j \phi_{k+1}(\xi),$$

which satisfies the orthogonal property (29a). Hence we have

$$(89) \quad (w - Pw)|_{I_j} = \sum_{i=k+2}^{\infty} \omega_i^j \phi_i(\xi) + (\omega_k^j - a_k^j) \phi_k(\xi) + (\omega_{k+1}^j - a_{k+1}^j) \phi_{k+1}(\xi).$$

Let Q_k be the standard L^2 -projection from $L^2(I)$ onto $P^k(I)$ with

$$Q_k \tilde{w}_j = \sum_{i=1}^{k+1} \omega_i^j \phi_i(\xi).$$

Then

$$\inf_{v \in P^k[-1,1]} \|\tilde{w}_j - v\| = \|\tilde{w}_j - Q_k \tilde{w}_j\|,$$

where $\tilde{w}_j - Q_k \tilde{w}_j = \sum_{i=k+2}^{\infty} \omega_i^j \phi_i(\xi)$ with $\|\phi_i\|^2 = 2/(2i-1)$ and $\phi_i(\pm 1) = (\pm 1)^{i+1}$, we obtain from the above expression the following bounds:

$$(90a) \quad \|\tilde{w}_j - \widetilde{P}w_j\|^2 = \|\tilde{w}_j - Q_k \tilde{w}_j\|^2 + \sum_{i=k}^{k+1} |\omega_i^j - a_i^j|^2 \frac{2}{2i-1},$$

$$(90b) \quad (\tilde{w}_j - \widetilde{P}w_j)(+1) = \sum_{i=k+2}^{\infty} \omega_i^j + \sum_{i=k}^{k+1} (\omega_i^j - a_i^j),$$

$$(90c) \quad |(\tilde{w}_j - \widetilde{P}w_j)(\xi)| \leq |\tilde{w}_j - Q_k \tilde{w}_j|_{\infty} + \sum_{i=k}^{k+1} |\omega_i^j - a_i^j| \cdot |\phi_i|_{\infty}.$$

Step 2. In order to estimate the sum of the coefficient ω_i^j , following [11], we consider

$$\partial_{\xi} \tilde{w}_j(\xi) = \sum_{i=1}^{\infty} \beta_i^j \phi_i(\xi).$$

Integration of this yields

$$\tilde{w}_j(\xi) = \tilde{w}_j(-1) + \sum_{i=1}^{\infty} \beta_i^j \int_{-1}^{\xi} \phi_i(\nu) d\nu,$$

and for $i \geq 2$ using the identity

$$\int_{-1}^{\xi} \phi_i(\nu) d\nu = \frac{1}{2i-1} (\phi_{i+1}(\xi) - \phi_{i-1}(\xi)),$$

we obtain

$$\tilde{w}_j(\xi) = (\tilde{w}_j(-1) + \beta_1^j) \phi_1 + \sum_{i=2}^{\infty} \frac{\beta_{i-1}^j}{2i-3} \phi_i(\xi) - \sum_{i=1}^{\infty} \frac{\beta_{i+1}^j}{2i+1} \phi_i(\xi).$$

Comparing coefficients in the Legendre expressions, one can conclude that

$$\omega_i^j = \frac{\beta_{i-1}^j}{2i-3} - \frac{\beta_{i+1}^j}{2i+1}, \quad i \geq 2;$$

therefore

$$(91a) \quad \sum_{i=k+2}^{\infty} \omega_i^j = \frac{\beta_{k+1}^j}{2k+1} + \frac{\beta_{k+2}^j}{2k+3},$$

$$(91b) \quad \sum_{i=k+2}^{\infty} \omega_i^j (-1)^{i+1} = (-1)^{k+1} \left(\frac{\beta_{k+1}^j}{2k+1} - \frac{\beta_{k+2}^j}{2k+3} \right).$$

Since $\|\partial_{\xi} \tilde{w}_j\|_I^2 = \sum_{i=1}^{\infty} (\beta_i^j)^2 \frac{2}{2i-1}$, we then have

$$(92) \quad \left| \sum_{i=k+2}^{\infty} \omega_i^j \phi_i(\pm 1) \right|^2 \leq \frac{1}{2k+1} \|\partial_{\xi} \tilde{w}_j\|^2,$$

$$(93) \quad \sum_{i=2}^{k+1} |\omega_i^j|^2 \leq 2 \|\partial_{\xi} \tilde{w}_j\|^2.$$

Step 3. We now estimate the difference $\omega_i^j - a_i^j$ for $i = k, k + 1$. From two interface conditions (29b)-(29c) it follows that

$$\begin{aligned} \sum_{i=k}^{k+1} [\phi_i(1)a_i^j + \phi_i(-1)a_i^{j+1}] &= 2\{w\}_{j+1/2} - \sum_{i=1}^{k-1} [\phi_i(1)\omega_i^j + \omega_i^{j+1}\phi_i(-1)], \\ \sum_{i=k}^{k+1} [g_0(i)a_i^j + g_1(i)a_i^{j+1}] &= h\widehat{w}_{x_{j+1/2}} - \sum_{i=1}^{k-1} [g_0(i)\omega_i^j + g_1(i)\omega_i^{j+1}]. \end{aligned}$$

This can be expressed as

$$\begin{aligned} \sum_{i=k}^{k+1} [\phi_i(1)(a_i^j - \omega_i^j) + \phi_i(-1)(a_i^{j+1} - \omega_i^{j+1})] &= \widetilde{b}_1^j := 2\{w\}_{j+1/2} \\ &\quad - \sum_{i=1}^{k+1} [\phi_i(1)\omega_i^j + \omega_i^{j+1}\phi_i(-1)], \\ \sum_{i=k}^{k+1} [g_0(i)(a_i^j - \omega_i^j) + g_1(i)(a_i^{j+1} - \omega_i^{j+1})] &= \widetilde{b}_2^j := h\widehat{w}_{x_{j+1/2}} \\ &\quad - \sum_{i=1}^{k+1} [g_0(i)\omega_i^j + g_1(i)\omega_i^{j+1}]. \end{aligned}$$

Since this system is uniquely solvable, using periodicity we obtain

$$(94) \quad \sum_{j=1}^N \sum_{i=k}^{k+1} |a_i^j - \omega_i^j|^2 \leq C \sum_{j=1}^N ((\widetilde{b}_1^j)^2 + (\widetilde{b}_2^j)^2).$$

From

$$\widetilde{b}_1^j = \sum_{i=k+2}^{\infty} [\phi_i(1)\omega_i^j + \omega_i^{j+1}\phi_i(-1)] = \sum_{i=k+2}^{\infty} \omega_i^j + \sum_{i=k+2}^{\infty} \omega_i^{j+1}(-1)^{i+1}$$

and (92) we have

$$\sum_{j=1}^N |\widetilde{b}_1^j|^2 \leq \sum_{j=1}^N \frac{2}{2k+1} (\|\partial_\xi \widetilde{w}_j\|^2 + \|\partial_\xi \widetilde{w}_{j+1}\|^2) \leq \frac{4}{2k+1} \sum_{j=1}^N \|\partial_\xi \widetilde{w}_j\|^2.$$

We rewrite \widetilde{b}_2^j as

$$\begin{aligned} \widetilde{b}_2^j &= \beta_0(\widetilde{w}_{j+1}(-1) - \widetilde{w}_j(1)) - \sum_{i=1}^{k+1} [g_0(i)\omega_i^j + g_1(i)\omega_i^{j+1}] \\ &\quad + (\partial_\xi \widetilde{w}_{j+1}(-1) + \partial_\xi \widetilde{w}_j(1)) + 4\beta_1 (\partial_\xi^2 \widetilde{w}_{j+1}(-1) - \partial_\xi^2 \widetilde{w}_j(1)). \end{aligned}$$

The terms in the first line reduce to

$$\beta_0 \sum_{i=k+2}^{\infty} \omega^{j+1}(-1)^{i+1} - \beta_0 \sum_{i=k+2}^{\infty} \omega^j - \sum_{i=2}^{k+1} [(g_0(i) + \beta_0)\omega_i^j + (g_1(i) + \beta_0(-1)^i)\omega_i^{j+1}].$$

Note that here the $i = 1$ term in the sum vanishes since $g_0(1) = -\beta_0$ and $g_1(1) = (-1)^i g_0(1)$. Based on all terms in \widetilde{b}_2^j , using (92), (93) and $\|v\|_{\infty, [-1, 1]}^2 \leq 2\|v\|_{1, [-1, 1]}^2$,

we obtain

$$\begin{aligned} \sum_{j=1}^N (\tilde{b}_2^j)^2 &\leq 8 \sum_{j=1}^N \left(\beta_0^2 \left| \sum_{i=k+2}^{\infty} \omega^j \phi_i(\pm 1) \right|^2 + k(\max_i |g_0(i)| + \beta_0)^2 \sum_{i=2}^{k+1} (\omega_i^j)^2 \right. \\ &\quad \left. + |\partial_\xi \tilde{w}_j|^2 + 16\beta_1^2 |\partial_\xi^2 \tilde{w}_j|^2 \right) \\ &\leq 8 \sum_{j=1}^N \left(\left(\frac{\beta_0^2}{2k+1} + 2k(\max_i |g_0(i)| + \beta_0)^2 \right) \|\partial_\xi \tilde{w}_j\|^2 \right. \\ &\quad \left. + 2\|\partial_\xi \tilde{w}_j\|_1^2 + 32\beta_1^2 \|\partial_\xi^2 \tilde{w}_j\|_1^2 \right) \\ &\leq C(k, \beta_0, \beta_1) \sum_{j=1}^N \|\partial_\xi \tilde{w}_j\|_{\min\{k, 2\}}^2. \end{aligned}$$

This is valid for $k \geq 1$ since when $k = 1$, $\beta_1 = 0$ is the default value.

These, when inserted into (94), yield

$$(95) \quad \sum_{j=1}^N \sum_{i=k}^{k+1} |a_i^j - \omega_i^j|^2 \leq C \sum_{j=1}^N \|\partial_\xi \tilde{w}_j\|_k^2.$$

Step 4. We finally complete the desired estimates. Inserting (92), (93) and (95) into (90), we obtain

$$\begin{aligned} \sum_{j=1}^N \|w - Pw\|_{I_j}^2 &\leq \frac{h}{2} \sum_{j=1}^N \|\tilde{w}_j - Q_k \tilde{w}_j\|^2 + C \frac{h}{2} \sum_{j=1}^N \|\partial_\xi \tilde{w}_j\|_k^2, \\ \sum_{j=1}^N |(w - Pw)_{j+1/2}|^2 &\leq C \sum_{j=1}^N \|\partial_\xi \tilde{w}_j\|_k^2, \\ \sum_{j=1}^N \|w - Pw\|_{\infty, I_j}^2 &\leq C \left(\sum_{j=1}^N \|\tilde{w}_j - Q_k \tilde{w}_j\|_\infty^2 + \sum_{j=1}^N \|\partial_\xi \tilde{w}_j\|_k^2 \right). \end{aligned}$$

Replacing w in these inequalities by $w - v$, where v is an arbitrary element in \mathbb{V}_h^k , and taking into account that $Q_k \tilde{v}_j = \tilde{v}_j$ gives

$$(96a) \quad \sum_{j=1}^N \|w - Pw\|_{I_j}^2 \leq \frac{h}{2} \sum_{j=1}^N \|\tilde{w}_j - Q_k \tilde{w}_j\|^2 + C \frac{h}{2} \sum_{j=1}^N \|\partial_\xi \tilde{w}_j - \partial_\xi \tilde{v}_j\|_k^2,$$

$$(96b) \quad \sum_{j=1}^N |(w - Pw)_{j+1/2}|^2 \leq C \sum_{j=1}^N \|\partial_\xi \tilde{w}_j - \partial_\xi \tilde{v}_j\|_k^2,$$

$$(96c) \quad \sum_{j=1}^N \|w - Pw\|_{\infty, I_j}^2 \leq C \left(\sum_{j=1}^N \|\tilde{w}_j - Q_k \tilde{w}_j\|_\infty^2 + \sum_{j=1}^N \|\partial_\xi \tilde{w}_j - \partial_\xi \tilde{v}_j\|_k^2 \right).$$

The standard projection estimates give

$$\|\tilde{w}_j - Q_k \tilde{w}_j\| \leq C |\tilde{w}_j|_{k+1}, \quad \|\tilde{w}_j - Q_k \tilde{w}_j\|_\infty \leq Ch^{k+1/2} |w|_{k+1, I_j} = Ch^{-1/2} |\tilde{w}_j|_{k+1}.$$

By Theorem 3.1.1 in [13], there exists a constant C such that

$$\inf_{q \in P^{k-1}[-1,1]} \|\partial_\xi \tilde{w}_j - q\|_k \leq C |\partial_\xi \tilde{w}_j|_k = C |\tilde{w}_j|_{k+1}.$$

We now simply insert these estimates into (96) to conclude that

$$\begin{aligned} \sum_{j=1}^N \|w - Pw\|_{I_j}^2 &\leq C \frac{h}{2} \sum_{j=1}^N |\tilde{w}_j|_{k+1}^2, \\ \sum_{j=1}^N |(w - Pw)_{j+1/2}|^2 &\leq C \sum_{j=1}^N |\tilde{w}_j|_{k+1}^2, \\ \sum_{j=1}^N |w - Pw|_{\infty, I_j}^2 &\leq C \sum_{j=1}^N (h^{-1} |\tilde{w}_j|_{k+1}^2 + |\tilde{w}_j|_{k+1}^2) \leq Ch^{-1} \sum_{j=1}^N |\tilde{w}_j|_{k+1}^2 \end{aligned}$$

for any $k \geq 1$. The corresponding estimate at $x_{j-1/2}$ is obtained by symmetry. From these estimates we obtain by standard scaling the estimates as claimed. \square

In order to extend the above estimates to the multi-dimensional projection we need the following estimates which follow easily from the above analysis.

Corollary 7.2. *Assume that $w|_{I_j} \in H^{s+1}(I_j)$. Then we have*

$$(97) \quad \|Pw\|^2 \leq C \sum_{l=0}^{\min\{k,2\}} h^{2l} \sum_{j=1}^N |w|_{l, I_j}^2$$

and

$$(98) \quad \|\partial_x^m (w - Pw)\|^2 \leq Ch^{2(p+1-m)} \sum_{j=1}^N |w|_{p+1, I_j}^2$$

for any $1 \leq p \leq \min\{s, k\}$.

Proof. From (87) and (88) it follows that

$$\|w\|^2 = \sum_{j=1}^N \|w\|_{I_j}^2 = \frac{h}{2} \sum_{j=1}^N \sum_{i=1}^\infty (\omega_i^j)^2 \frac{2}{2i-1}$$

and

$$\|Pw\|^2 = \sum_{j=1}^N \|Pw\|_{I_j}^2 = \frac{h}{2} \sum_{j=1}^N \left(\sum_{i=1}^{k+1} (\omega_i^j)^2 \frac{2}{2i-1} + \sum_{l=k}^{k+1} |a_l^j - \omega_l^j|^2 \frac{2}{2l-1} \right).$$

Hence using (95) we have

$$\|Pw\|^2 \leq \|w\|^2 + \frac{Ch}{2} \sum_{j=1}^N \|\partial_\xi \tilde{w}_j\|_{\min\{k,2\}}^2,$$

which by a rescaling gives (97).

From (89) we have

$$(99) \quad \begin{aligned} \partial_x^m (w - Pw)|_{I_j} &= \left(\partial_\xi^m (\tilde{w}_j - Q_k \tilde{w}_j) + (\omega_k^j - a_k^j) \phi_k^{(m)}(\xi) \right. \\ &\quad \left. + (\omega_{k+1}^j - a_{k+1}^j) \phi_{k+1}^{(m)}(\xi) \right) \left(\frac{h}{2} \right)^{-m}. \end{aligned}$$

We continue as in the proof of Theorem 7.1 for the case $m = 0$; finally, a standard scaling leads to (98). \square

Theorem 7.3. *Let $P : H^{s+1} \rightarrow \mathbb{V}_h^k$ denote the one-dimensional global projection defined by (29) and denote by $\Pi = P^{(x)} \otimes P^{(y)}$ its tensor product along the two dimensions (or (69) with $d = 2$). Then Π is well defined on Ω . If $w \in H^{s+1}(\Omega)$ for some $s \geq 1$, then*

$$(100a) \quad \|w - \Pi w\| \leq Ch^{p+1}|w|_{p+1,\Omega},$$

$$(100b) \quad \|w - \Pi w\|_{\Gamma_h} \leq Ch^{p+1/2}|w|_{p+1,\Omega},$$

$$(100c) \quad \|w - \Pi w\|_\infty \leq Ch^{p-1/2}|w|_{p+1,\Omega}$$

for any $1 \leq p \leq \min\{s, k\}$.

Proof. By the definition of the tensor product, we see that $(P^{(y)}w)(x, y)$ is a polynomial of degree k in y with x -dependent coefficients that belong to $H^{s+1}(0, L_x)$; hence $\Pi w = P^{(x)}(P^{(y)}w)$ is well defined and also

$$\Pi w = P^{(y)}(P^{(x)}w).$$

To prove the error estimate (100a), we use (97), (98) through

$$\|w - \Pi w\| \leq \|w - P^{(x)}w\| + \|P^{(x)}(w - P^{(y)}w)\|.$$

We bound the first term using (98) with $m = 0$ to obtain

$$\|w - P^{(x)}w\|^2 \leq Ch^{2(p+1)} \sum_{K_{ij}} \int_{K_{ij}} |\partial_x^{p+1} w|^2 dx dy.$$

Here and below \sum denotes the sum over all $1 \leq i \leq N_x, 1 \leq j \leq N_y$, unless index is explicitly specified.

From (97) we have

$$\|P^{(x)}w\|^2 \leq C \sum_{l=0}^{\min\{k,2\}} h^{2l} \sum_{K_{ij}} \int_{K_{ij}} |\partial_x^l w|^2 dx dy,$$

with which the second term is bounded by

$$\begin{aligned} \|P^{(x)}(w - P^{(y)}w)\|^2 &\leq C \sum_{l=0}^{\min\{k,2\}} h^{2l} \sum_{K_{ij}} \int_{K_{ij}} |\partial_x^l (w - P^{(y)}w)|^2 dx dy \\ &= C \sum_{l=0}^{\min\{k,2\}} h^{2l} \sum_{K_{ij}} \int_{K_{ij}} |\partial_x^l w - P^{(y)}\partial_x^l w|^2 dy dx \\ &\leq C \sum_{l=0}^{\min\{k,2\}} h^{2l} h^{2(t+1)} \sum_{K_{ij}} \int_{K_{ij}} |\partial_y^{t+1} \partial_x^l w|^2 dx dy \end{aligned}$$

for any $0 \leq t \leq p - l$. Selecting $t = p - l$ gives

$$\|P^{(x)}(w - P^{(y)}w)\|^2 \leq Ch^{2(p+1)} \sum_{l=0}^{\min\{k,2\}} \sum_{K_{ij}} \int_{K_{ij}} |\partial_x^l \partial_y^{p-l+1} w|^2 dx dy.$$

Thus (100a) follows. In order to prove (100b) we write

$$(101) \quad \|w - \Pi w\|_{\Gamma_h}^2 = \sum \int_{J_j} |w - \Pi w|_{x_{i+1/2}}^2 dy + \sum \int_{I_i} |w - \Pi w|_{y_{j+1/2}}^2 dx.$$

In virtue of (85) and (97), the first term on the right is bounded from above by

$$\begin{aligned} & 2 \sum \int_{J_j} |w - P^{(x)}w|_{x_{i+1/2}}^2 dy + 2 \sum \int_{J_j} |P^{(x)}(w - P^{(y)}w)|_{x_{i+1/2}}^2 dy \\ & \leq 2Ch^{2p+1} \sum \int_{K_{ij}} |\partial_x^{p+1}w|^2 dx dy + 2C \sum \sum_{l=0}^{\min\{k,2\}} h^{2l} \int_{I_i} |\partial_x^l(w - P^{(y)}w)|_{y_{j+1/2}}^2 dx \\ & \leq 2Ch^{2p+1} \sum \int_{K_{ij}} |\partial_x^{p+1}w|^2 dx dy + 2C \sum \sum_{l=0}^{\min\{k,2\}} h^{2l} h^{2(t+1)} \int_{K_{ij}} |\partial_y^{t+1}\partial_x^l w|^2 dx dy. \end{aligned}$$

Selecting $t = p - l$ gives

$$\sum \int_{J_j} |w - \Pi w|_{x_{i+1/2}}^2 dy \leq Ch^{2p+1}|w|_{p+1,\Omega}^2.$$

The second term in (101) shares this bound as well. Hence (100b) is justified.

Finally we prove (100c), starting from

$$(102) \quad \sum |w - \Pi w|_{\infty,K_{ij}}^2 \leq 2 \sum |w - P^{(x)}w|_{\infty,K_{ij}}^2 + 2 \sum |P^{(x)}(w - P^{(y)}w)|_{\infty,K_{ij}}^2.$$

Rescaling the Sobolev estimate $|\tilde{v}|_{\infty,[-1,1]}^2 \leq 2\|\tilde{v}\|_{1,[-1,1]}^2$ we have

$$\sum_{j=1}^{N_y} |v(y)|_{\infty,J_j}^2 h \leq 2 \sum_{l=0}^1 h^{2l} \|\partial_y^l v\|_{0,[0,L_y]}^2.$$

Note that if $y \in \mathbb{R}^{d-1}$, the factor h on the left should be h^{d-1} and the summation index on the right runs up to $q > \frac{d-1}{2}$. With this, (86), and (98) with $m = 0$, estimating the first term in (102) gives

$$\begin{aligned} \sum |w - P^{(x)}w|_{\infty,K_{ij}}^2 & \leq 2Ch^{2t_1} \sum \int_{I_i} |\partial_x^{t_1+1}w|_{\infty,J_j}^2 dx \\ & \leq 2Ch^{2t_1} \sum \int_{K_{ij}} h^{-1} \sum_{l=0}^1 h^{2l} |\partial_y^l \partial_x^{t_1+1}w|^2 dx dy \\ & \leq Ch^{2p-1}|w|_{p+1,\Omega}^2, \end{aligned}$$

where we have selected $t_1 = p - l$.

Estimating the second term with (97) and (98) with $m = 0$ yields

$$\begin{aligned} 2 \sum |P^{(x)}(w - P^{(y)}w)|_{\infty,K_{ij}}^2 & \leq 2C \sum \sum_{l=0}^{\min\{k,2\}} h^{2l} \int_{I_i} |\partial_x^l(w - P^{(y)}w)|_{\infty,J_j}^2 dx \\ & = 2C \sum \sum_{l=0}^{\min\{k,2\}} h^{2l} \int_{I_i} |\partial_x^l w - P^{(y)}(\partial_x^l w)|_{\infty,J_j}^2 dx \\ & \leq 2C \sum_{l=0}^{\min\{k,2\}} h^{2l} \sum \int_{K_{ij}} h^{2t} |\partial_y^{t+1}\partial_x^l w|^2 dx dy \\ & \leq Ch^{2p}|w|_{p+1,\Omega}^2, \end{aligned}$$

where we have selected $t = p - l$.

Therefore,

$$\sum |w - \Pi w|_{\infty, K_{ij}}^2 \leq Ch^{2p-1} |w|_{p+1, \Omega}^2. \quad \square$$

Remark 7.1. Note that the above analysis when applied to the d -dimensional case yields the following estimate:

$$\sum |w - \Pi w|_{\infty, K_\alpha}^2 \leq Ch^{2p+1-d} |w|_{p+1, \Omega}^2.$$

Hence, for $d > 2$, (100c) needs to be replaced by

$$(103) \quad \|w - \Pi w\|_{\infty} \leq Ch^{p+1/2-d/2} |w|_{p+1, \Omega},$$

while (100a) and (100b) remain valid; these together lead to (71) as needed.

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