Problem 9. If $\theta \neq \frac{\pi}{2} + \pi k,\ k = 0, \pm 1, \pm 2, \pm 3, \ldots,$ then $\tan\theta$ is defined and $\sin^2\theta \leq \tan^2\theta$. In fact for such θ and any positive integer n,

$$\sin^2\theta + \sin^4\theta + \sin^6\theta + \cdots + \sin^{2n}\theta \leq \tan^2\theta.$$

Prove this inequality.

Solution 9. Because $x \neq \dfrac{\pi}{2} + \pi k$, $\sin^2 x < 1$. Therefore this finite geometric sum sums to

$$\sin^2\theta+\sin^4\theta+\sin^6\theta+\cdots+\sin^{2n}\theta = \frac{\sin^2 - \sin^{2n+2} x}{1-\sin^2 x} \leq \frac{\sin^2 x}{1-\sin^2 x} = \frac{\sin^2 x}{\cos^2 x} = \tan^2 x.$$