Problem 6. What is the maximum number of points with integer coordinates that can lie on a circle of center $(\sqrt{2}, \sqrt{3})$?

Solution 6. There can be at most one point with integer coordinates on any circle with center $(\sqrt{2}, \sqrt{3})$.

Assume that there are two points, $A = (x_1, y_1)$ and $B = (x_2, y_2)$ with integer coordinates and on a circle centered at $(\sqrt{2}, \sqrt{3})$. The midpoint of $\overline{AB}$ is the point

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = (m_1, m_2)$$

is a point with rational coordinates. The line through M and perpendicular to $\overline{AB}$ passes through the center $O = (\sqrt{2}, \sqrt{3})$ of the circle. The slope of this line can be expressed in two ways:

$$\frac{m_2 - \sqrt{3}}{m_1 - \sqrt{2}} = \text{slope of } \overline{MO} = -\frac{1}{\text{slope of } \overline{AB}} = -\frac{x_2 - x_1}{y_2 - y_1}.$$

In this last expression, m_1, m_2 and $-\frac{x_2-x_1}{y_2-y_1} = r$ are all rational numbers. We prove that this is impossible. In particular, with a little algebra

$$\frac{m_2 - \sqrt{3}}{m_1 - \sqrt{2}} = r \quad \text{implies} \quad \sqrt{3} = m_2 - m_1 r + r\sqrt{2}.$$

Squaring both sides gives

$$3 = (m_2 - m_1 r)^2 + 2r(m_2 - m_1 r)\sqrt{2} + 2r^2 \quad \text{and then} \quad \sqrt{2} = \frac{3 - (m_2 - m_1 r)^2 - 2r^2}{2r(m_2 - m_1 r)}.$$

The last expression is impossible because $\sqrt{2}$ is irrational and the expression on the right is rational.