Problem 5. Prove that for *all* positive integers n and *all* real numbers x,

$$\left\lfloor \frac{\lfloor nx \rfloor}{n} \right\rfloor = \lfloor x \rfloor.$$

(If a is a real number, then $\lfloor a \rfloor$ is equal to the greatest integer less than or equal to x. For example,

$$\lfloor 3.14 \rfloor = 3, \qquad \lfloor 4 \rfloor = 4, \qquad \lfloor -3.14 \rfloor = -4.)$$

Solution 5. Because $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$, there is an ϵ, $0 \leq \epsilon < 1$ with

$$x = \lfloor x \rfloor + \epsilon.$$

Then

$$nx = n\lfloor x \rfloor + n\epsilon,$$

and

$$\lfloor nx \rfloor = \lfloor n\lfloor x \rfloor + n\epsilon \rfloor = n\lfloor x \rfloor + \lfloor n\epsilon \rfloor.$$

Because $\lfloor n\epsilon \rfloor < n$, we have

$$\left\lfloor \frac{\lfloor n\epsilon \rfloor}{n} \right\rfloor = 0.$$

Therefore

$$\left\lfloor \frac{\lfloor nx \rfloor}{n} \right\rfloor = \left\lfloor \frac{n\lfloor x \rfloor + \lfloor n\epsilon \rfloor}{n} \right\rfloor = \lfloor x \rfloor + \left\lfloor \frac{\lfloor n\epsilon \rfloor}{n} \right\rfloor = \lfloor x \rfloor.$$