**Problem 3.** Parallel lines $\ell$ and $m$ and a point $P$ are in the plane. Point $P$ lies between $\ell$ and $m$ and is 3 units from $\ell$ and 7 units from $m$. A right triangle is constructed with one vertex on $\ell$, one vertex on $m$, and right angle at $P$. What is the minimum possible area of this triangle?

**Solution 3.** In both cases the minimum area is 21.

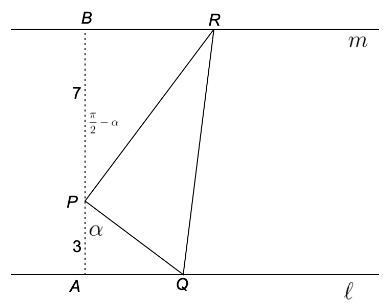
We first consider the case in which $P$ is between $\ell$ and $m$. Draw the line through $P$ and perpendicular to $\ell$ abd $m$ and let this line intersect $\ell$ at point $A$ and intersect $m$ at $B$. Consider a right triangle with right angle at $P$ and vertex $Q$ on $\ell$ and $R$ on $m$. Let $\angle APQ = \alpha$. Then $\angle BPR = \frac{\pi}{2} - \alpha$.

We then have

$$PQ = 3 \sec \alpha$$

and

$$PR = 7 \sec\left(\frac{\pi}{2} - \alpha\right) = 7 \csc \alpha.$$

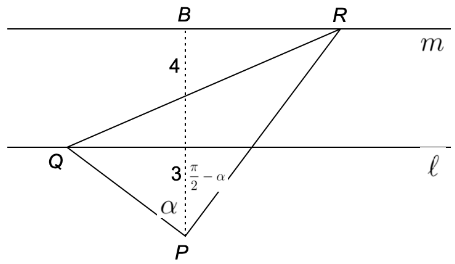



Therefore

$$\text{Area}(PQR) = \frac{1}{2} PQ \cdot PR = \frac{21}{2 \sin \alpha \cos \alpha} = \frac{21}{\sin(2\alpha)}.$$

This expression is minimized when $\sin(2\alpha)$ is as large as possible, that is, when $\sin(2\alpha) = 1$, which occurs when $\alpha = \frac{\pi}{2}$. For this choice of $\alpha$, the area assumes it minimum value of 21.

Next consider the case in which $P$ is not between $\ell$ and $m$. Referring to the similarly labeled Figure below, Once again,




$$\text{Area}(PQR) = \frac{1}{2}PQ \cdot PR = \frac{1}{2}(3\sec\alpha)\left(7\sec\left(\frac{\pi}{2}-\alpha\right)\right) = \frac{21}{\sin(2\alpha)}.$$

The minimum area of 21 is again acheived by taking $\alpha = \frac{\pi}{4}$.