Problem 2. Find all sets of consecutive integers that sum to 2020.

Solution 2. There are eight such sequences, four of which involve negative integers. First consider the case in which

$$-m+(-(m-1))+(-(m-2))+\cdots+(-1)+0+1+\cdots+m+(m+1)+\cdots+n=2020,$$

where m and n are positive integers. For this sum to be 2020, it is clear that $n > m$ and that (after cancellation)

$$(m+1)+(m+2)+\cdots+n=2020.$$

Thus it suffices to consider the case in which all integers in the sequence are positive. From each such sequence we can obtain a companion sequence with some negative elements.
It is well known that for positive integer N,

$$1+2+\cdots+N=\frac{N(N+1)}{2}.$$

Therefore if m and n are positive integers with $m < n$ and

$$(m+1)+(m+2)+\cdots+n=2020,$$

we have

$$\frac{n(n+1)}{2}-\frac{m(m+1)}{2}=2020.$$

With a little algebra we find

$$4040=(n^2-m^2)+(n-m)=(n-m)(n+m+1).$$

Because $n-m$ and $n+m+1$ are numbers of different parity, $n-m$ and $n+m+1$ are positive factors of 4040 with one factor even and the other factor odd. The number 4040 can be factored into factors of different parity in four ways,

$$4040=1\cdot 4040=5\cdot 808=101\cdot 40=505\cdot 8.$$

Since $n+m+1>n-m$, we have four possibilities:

$n+m+1=4040$, and $n-m=1$ leading to $n=2020$ and $m=2019$,

$$n + m + 1 = 808, \quad \text{and} \quad n - m = 5 \qquad \text{leading to} \qquad n = 406 \quad \text{and} \quad m = 401,$$

$$n + m + 1 = 101, \quad \text{and} \quad n - m = 40 \qquad \text{leading to} \qquad n = 70 \quad \text{and} \quad m = 30,$$

and

$$n + m + 1 = 505, \quad \text{and} \quad n - m = 8 \qquad \text{leading to} \qquad n = 256 \quad \text{and} \quad m = 248.$$

Letting $[j,\, k]$ denote the set of integers $\{j,\, j+1, \ldots, k\}$, we have found that the following sets of positive integers sum to 2020:

$$[2020,\, 2020], \qquad [402,\, 406], \qquad [249, 256] \qquad [31, 70].$$

If we allow negative integers, we have the additional four sets

$$[-2019,\, 2020], \qquad [-401,\, 406], \qquad [-248,\, 256] \qquad [-30,\, 70].$$