**Problem 14.** Prove that there are infinitely many pairs $(x, y)$ of rational numbers such that

$$x^2 + y^2 = x^3 + y^3,$$

then describe all of the solutions.

**Solution 14.** Let $(x, y)$ be a rational solution to the given equation. If $x \neq 0$, then there is a rational number $m$ with $y = mx$. Then

$$x^2(1 + m^2) = x^3(1 + m^3) \qquad \text{from which} \qquad x = \frac{1 + m^2}{1 + m^3}.$$

Therefore for any rational $m \neq -1$, the pair

$$x = \frac{1 + m^2}{1 + m^3}, \quad y = m\frac{1 + m^2}{1 + m^3}$$

is a rational solution to $x^2 + y^2 = x^3 + y^3$. Furthermore, every rational solution $(x, y)$ with $x \neq 0$ and $y \neq -x$ is of this form.

It is easy to check that if $x = 0$ then $y = 0$ or $y = 1$, and that if $y = -x$ (the $m = -1$ case) then $x = y = 0$. Therefore the rational solutions are

$$(0,\, 0), \quad (0,\, 1), \quad \text{and} \quad \left(\frac{1 + m^2}{1 + m^3},\, m\frac{1 + m^2}{1 + m^3}\right), \quad m \text{ rational with } m \neq -1.$$