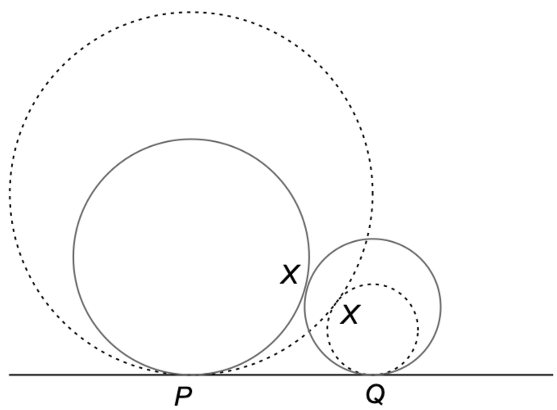
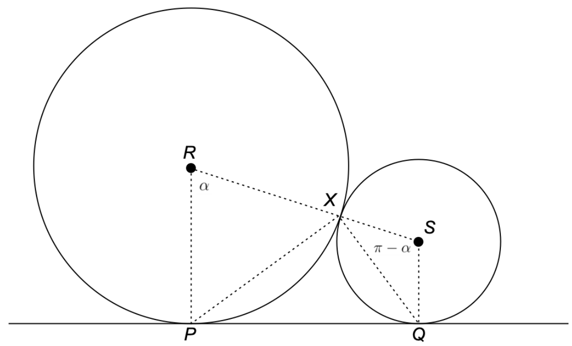
Problem 13. Let ℓ be a line in the plane and let P and Q be distinct points on ℓ. Circles C_1 aned C_2 are tangent to ℓ at P and Q respectively, and tangent to each other at the point X. The set of all possible points X that arise in this way is a curve consisting of several points. Describe this curve. Two examples of possible points X are shown in the figure below.



Solution 13. The curve is a circle with diameter $\overline{PQ}$. To prove this we show that for any two tangent circles, $\angle PXQ$ is a right angle. (Recall that a point X is on a circle of diameter $\overline{PQ}$ if and only if $\angle PXQ$ is a right angle.

Consider a pair of circles generating a point X. Let R be the center of the circle tangent to ℓ at P and S the center of the circle tangent to ℓ at Q. Then R, X, and S are collinear. Because $RPQS$ is a trapezoid with right angles at P and Q, angles PRX and QSX are supplementary: let $\angle PRX = \alpha$, so $\angle QSX = \pi - \alpha$.

Because $RP = RX$ and $SQ = SX$, it follows that $\angle RXP = \frac{\pi}{2} - \frac{\alpha}{2}$ and



$\angle SXQ = \frac{\alpha}{2}$. Because these two angles are complementary, $\angle PXQ$ is a right angle. Therefor X lies on the circle with diameter $\overline{PQ}$.

Some solver considered only circles on one side of ℓ and deduced that the curve was a half circle. These solutions were considered correct.