

Problem 12. Let p and q be (positive) prime numbers. Find all pairs (x, y) of positive integer solutions to the equation

$$\frac{p}{x} + \frac{q}{y} = 1.$$

Solution 12. From

$$\frac{p}{y} + \frac{q}{x} = 1 \quad \text{we have} \quad py + qx = xy \quad \text{and then} \quad xy - qx - py + pq = pq.$$

Factoring the left side of the last equation gives

$$(x - p)(y - q) = pq.$$

Because x and y are positive integers and p and q are prime we must have

$$x - p = p \quad \text{and} \quad y - q = q \quad \text{leading to} \quad (x, y) = (2p, 2q)$$

or

$$x - p = q \quad \text{and} \quad y - q = p \quad \text{leading to} \quad (x, y) = (p + q, p + q)$$

or

$$x - p = 1 \quad \text{and} \quad y - q = pq \quad \text{leading to} \quad (x, y) = (p + 1, pq + q)$$

or

$$x - p = pq \quad \text{and} \quad y - q = 1 \quad \text{leading to} \quad (x, y) = (pq + p, q + 1).$$

If $p = q$, then the first two solutions are the same and we have three distinct solution sets. If $p \neq q$, then the four possibilities result in four different solution sets.