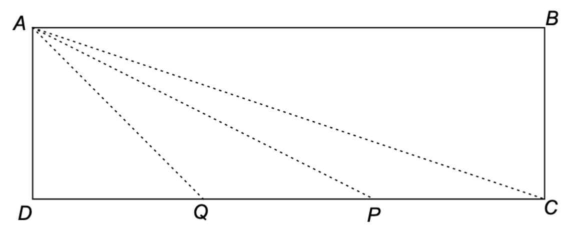
Problem 10. Let $ABCD$ be a rectangle with $AB = 3BC$. Points P and Q are on CD with Q between P and D, and

$$DQ = QP = PC,$$

as shown below. Prove that

$$\angle AQD = \angle APD + \angle ACD.$$



Solution. Note that $0 \leq \angle AQD,\ \angle APD,\ \angle ACD \leq \dfrac{\pi}{4}$ and that

$$\tan(\angle ACD) = \frac{1}{3}, \quad \tan(\angle APD) = \frac{1}{2}, \quad \text{and} \quad \tan(\angle AQD) = 1.$$

Then

$$\begin{aligned}\tan(\angle APD + \angle ACD) =& \frac{\tan(\angle APD) + \tan(\angle ACD)}{1 - (\tan(\angle APD))(\tan(\angle ACD))} \\ =& \frac{\frac{1}{3} + \frac{1}{2}}{1 - \frac{1}{3} \cdot \frac{1}{2}} = 1 = \tan(\angle AQD).\end{aligned}$$

Becasue all angles involved have measures in the interval $\left(0, \frac{\pi}{2}\right)$, it must be the case that

$$\angle APD + \angle ACD = \frac{\pi}{4} = \angle APD.$$