Solid of Revolution: Complicated Example

33 minutes ago by admin

Recall that we have a solid of revolution defined by rotating about the x-axis the region where

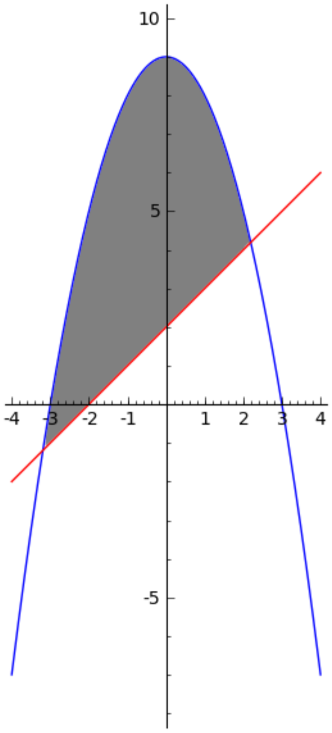
$$x + 2 \le y \le 9 - x^2$$

```
x,y = var('x','y')

R = 9 - x^2
r = x + 2

R,r
```

$\left(-x^2 + 9, x + 2\right)$

We can plot this xy-region to see what it looks like.

```
P = plot(R(x=x), (x,-4,4), color="blue") + plot(r(x=x), (x,-4,4), color="red") + region_plot( [R(x=x) >= y, r(x=x) <= y], (-4,4), (-8,10), incol="gray")


P.set_aspect_ratio(1.0)
P
```



An important step to do is to compute the bounds here. This can be done by setting $R(x) = r(x)$ and solving for x.

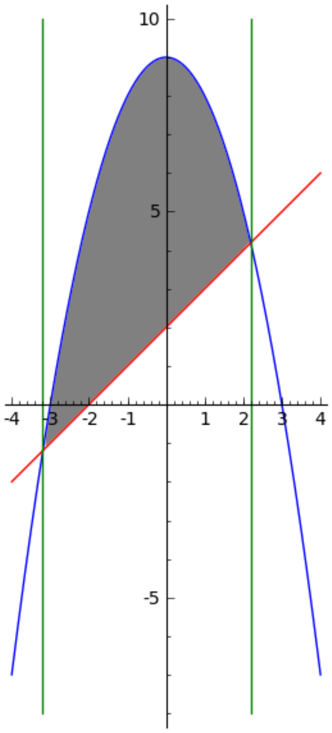
Note: Due to a mistake I made in my notes, this was harder than I anticipated.

```
sol = solve(R(x=x) == r(x=x), x)
a = sol[0].right_hand_side()
b = sol[1].right_hand_side()
sol
```

$\left[x = -\frac{1}{2}\sqrt{29} - \frac{1}{2}, x = \frac{1}{2}\sqrt{29} - \frac{1}{2}\right]$

Let's draw these boundary lines on the plot.

```
Q = P + line([(sol[0].right_hand_side(),-8), (sol[0].right_hand_side(), 10)], color="green") +
line([(sol[1].right_hand_side(),-8), (sol[1].right_hand_side(), 10)], color="green")
Q
```



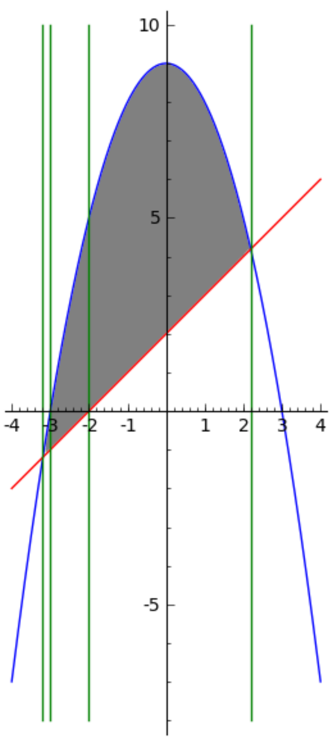
There is some problem with this!

In particular, the radius functions become negative! This violates our integration assumptions, so we should consider a few different parts of the region.

First, from $a = -\frac{\sqrt{29}-1}{2}$ to -3, both functions are nonpositive. Thus, we can use $\int_a^{-3} \pi\left[(x+2)^2 - (9-x^2)^2\right] dx$ to calculate the volume of this region. (Notice that $x+2$ becomes the "outer" radius here.)

From -2 to $b = \frac{\sqrt{29}-1}{2}$, both functions are nonnegative, so we can use the standard integral $\int_{-2}^{b} \pi\left[(9-x^2)^2 - (x+2)^2\right] dx$ to calculate the volume in this region.

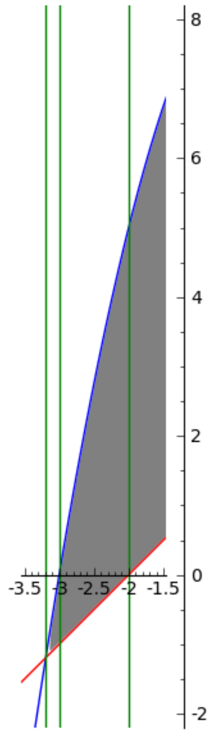
```
Q = P + line([(sol[0].right_hand_side(),-8), (sol[0].right_hand_side(), 10)], color="green") +
line([(sol[1].right_hand_side(),-8), (sol[1].right_hand_side(), 10)], color="green") + line([(-3,-8),(-3,10)],
color="green") + line([(-2,-8),(-2,10)], color="green")
Q
```



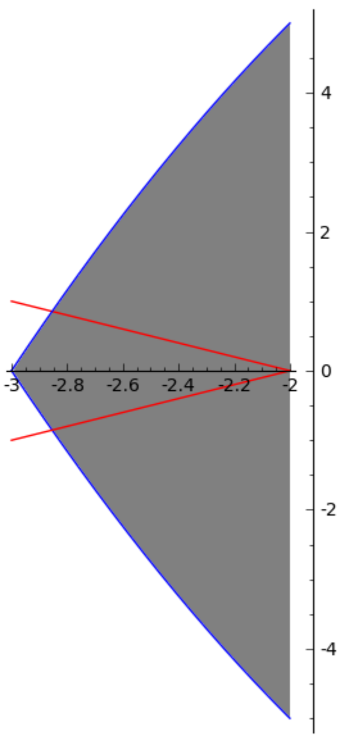
However, let's zoom in on the region between -3 and -2.

```
Q = P + line([(sol[0].right_hand_side(),-8), (sol[0].right_hand_side(), 10)], color="green") +
line([(sol[1].right_hand_side(),-8), (sol[1].right_hand_side(), 10)], color="green") + line([(-3,-8),(-3,10)],
color="green") + line([(-2,-8),(-2,10)], color="green")
```

```
Q.set_axes_range(-3.5,-1.5,-2,8)
Q
```




```
P1 = plot(R(x=x), (x,-3,-2), color="blue") + plot(r(x=x), (x,-3,-2), color="red")
P2 = plot(-R(x=x), (x,-3,-2), color="blue") + plot(-r(x=x), (x,-3,-2), color="red")
P3 = sage.plot.contour_plot.region_plot( [min(-R(x=x),r(x=x)) <= y, max(R(x=x),-r(x=x)) >= y], (-3,-2), (-5,5),
incol="gray",zorder=0)
Q = P2 + P1 + P3
Q.set_aspect_ratio(0.25)
Q
```



Notice above that the vertical reflection intersect in a point around -2.85. That is, where $9 - x^2 = -(x + 2)$.

Let's find this intersection point.

```
sol2 = solve(R(x=x) == -r(x=x), x)
c = sol2[0].right_hand_side()
sol2
```

$$\left[x = -\frac{3}{2}\sqrt{5} + \frac{1}{2}, x = \frac{3}{2}\sqrt{5} + \frac{1}{2}\right]$$

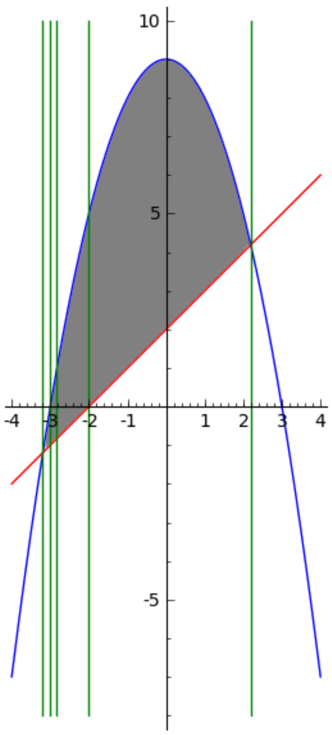
Let $c = -\frac{3\sqrt{5}-1}{2}$ be this intersection spot.

Now there are four distinct regions:

1. The region where both are nonpositive: $\int_a^{-3} \pi\left[(x+2)^2 - (9-x^2)^2\right]dx$. [Washer method, but reversed.]
2. The region where $x+2 \le 0, 9-x^2 \ge 0$, and $|x+2| \ge |9-x^2|$: $\int_{-3}^{c} \pi(x+2)^2dx$. [Disk method with $x+2$.]
3. The region where $x+2 \le 0, 9-x^2 \ge 0$, and $|x+2| \le |9-x^2|$: $\int_{-3}^{c} \pi(9-x^2)^2dx$. [Disk method with $9-x^2$.]
4. The region where both are nonnegative: $\int_{-2}^{b} \pi\left[(9-x^2)^2 - (x+2)^2\right]dx$. [Washer method, normal.]

See the plot below.

```
Q = P + line([(sol[0].right_hand_side(),-8), (sol[0].right_hand_side(), 10)], color="green") +
line([(sol[1].right_hand_side(),-8), (sol[1].right_hand_side(), 10)], color="green") + line([(-3,-8),(-3,10)],
color="green") + line([(-2,-8),(-2,10)], color="green")+ line([(sol2[0].right_hand_side(),-8),
(sol2[0].right_hand_side(),10)], color="green")
Q
```



We can now compute the four definite integrals, then sum them up to find the final answer.

```
from sage.symbolic.integration.integral import definite_integral

V1 = pi * definite_integral( (x+2)^2 - (9-x^2)^2, x,   a, -3)
V2 = pi * definite_integral( (x+2)^2,             x, -3,  c)
V3 = pi * definite_integral( (9-x^2)^2,           x,  c, -2)
V4 = pi * definite_integral( (9-x^2)^2 - (x+2)^2, x, -2,  b)
```

```
-1/15*pi*(319*sqrt(29) - 1720) -1/3*pi*(45*sqrt(5) - 101)
1/5*pi*(300*sqrt(5) - 625) 1/15*pi*(319*sqrt(29) + 1587)
```

```
V1.expand().simplify()
```

$\frac{344}{3}\pi - \frac{319}{15}\sqrt{29}\pi$

```
V2.expand().simplify()
```

$\frac{101}{3}\pi - 15\sqrt{5}\pi$

```
V3.expand().simplify()
```

$-125\pi + 60\sqrt{5}\pi$

```
V4.expand().simplify()
```

$\frac{529}{5}\pi + \frac{319}{15}\sqrt{29}\pi$

Here is our final volume of the solid.

```
(V1 + V2 + V3 + V4).expand().simplify()
```

$\frac{1937}{15}\pi + 45\sqrt{5}\pi$

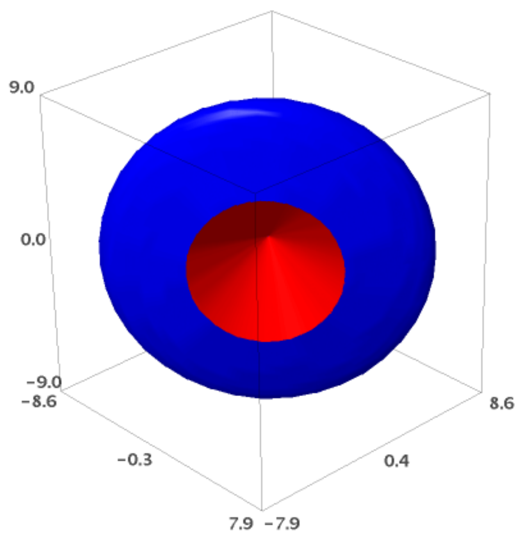
For some extra help visualizing what is happening, here are some 3D plots.

The first plot is looking down the x-axis, and we can see that the "inner cone" meets the x-axis (at $x = -2$)

```
P3D = revolution_plot3d(R(x=x), (x,a,b), color="blue", parallel_axis='x', alpha=0.5)+revolution_plot3d(r(x=x), (x,a,b),
color="red", parallel_axis='x', alpha=0.5)
```

```
Q1 = P3D.rotate((0,0,1),pi/4)
Q1.show(aspect_ratio=(1,1,1))
```

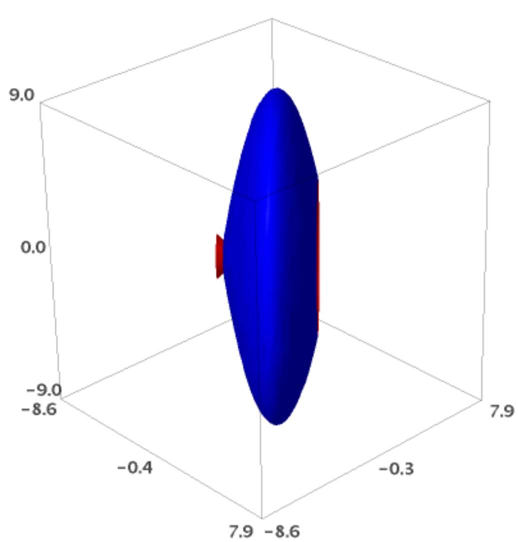
Sleeping... Make Interactive



This plot is the side-view that matches looking at the xy-plane.

```
Q2 = P3D.rotate((0,0,1), -pi/4)
Q2.show(aspect_ratio=(1,1,1))
```

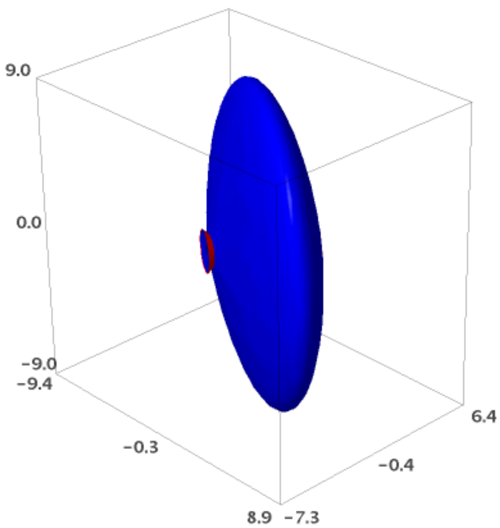
Sleeping... Make Interactive



This angle from a little more to the left shows the region where $|x+2| \geq |9-x^2|$, so the red band is the "outside".

```
Q3 = P3D.rotate((0,0,1), -pi/3)
Q3.show(aspect_ratio=(1,1,1))
```

Sleeping... Make Interactive


9.0
0.0
−9.0
−9.4
−0.3
8.9
−7.3
−0.4
6.4