Appendix A

Elementary Definitional Equivalence

An algebraizable logic can also be characterized in terms of the definitional equivalence of two elementary (i.e., first-order) theories, namely, the universal Horn theory ES discussed in Chapter 1.3 (a theory without equality), and the elementary theory of a quasivariety (a theory with equality). The treatment of elementary definitional equivalence we use here closely follows Tarski, Mostowski, and Robinson [45].

Let E_1 and E_2 be arbitrary elementary theories over the first-order languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Let R be a relation symbol of $\mathcal{L}_1$ that is not in $\mathcal{L}_2$. By a *possible definition* of R in $\mathcal{L}_2$ we mean a formula of the form

$$\forall p_0 \ldots p_{n-1}(Rp_0 \ldots p_{n-1} \leftrightarrow \alpha(p_0, \ldots, p_{n-1}))$$

where n is the rank of R and α is an arbitrary first-order formula of $\mathcal{L}_2$ with free variables $p_0, \ldots, p_{n-1}$. (For the purposes of the present discussion we treat operation symbols of rank n as relation symbols of rank $n+1$ in the usual way.) E_1 and E_2 are said to be *elementarily definitionally equivalent* if there exists a system Γ of possible definitions of the relation symbols of $\mathcal{L}_i \setminus \mathcal{L}_{1-i}$ in $\mathcal{L}_{1-i}$, for $i = 0, 1$, such that the theory E over the the combined language that is axiomatized by $E_1 \cup E_2 \cup \Gamma$ is a conservative extension of both E_1 and E_2; i.e., the intersection of E with the set of $\mathcal{L}_i$-sentences coincides with E_i.

The relationship between a deductive system $\mathcal{S}$ and a quasivariety K expressed in Definition 2.8 can be viewed as definitional equivalence exactly in this sense, but with the possible definitions required to be of a very special form. What is interesting here is that, in the event $\mathcal{S}$ is protoalgebraic, one can allow the possible definitions to be of a much more general form without getting a more general notion of equivalence.

Let $\mathcal{S}$ be an arbitrary deductive system over the propositional language $\mathcal{L}$, and let ES be its associated universal Horn theory over the first-order language (without equality) $\mathcal{L}_D$. Let K be any quasivariety over the first-order language $\mathcal{L}_\approx$ obtained by adjoining the equality symbol $\approx$ to $\mathcal{L}$. We write $E\mathsf{K}$ for the elementary (first-order) theory of K.

Recall from Chapter 1.4.1 that a deductive system $\mathcal{S}$ is ***protoalgebraic*** if the Leibniz equivalence operator $\boldsymbol{\Omega}$ is order-preserving on the lattice of $\mathcal{S}$-theories.

Theorem A.1 *Let $\mathcal{S}$ be a deductive system and* K *a quasivariety.*

(i) *If $\mathcal{S}$ is algebraizable with equivalent semantics* K, *then $E\mathcal{S}$ and EK are elementarily definitionally equivalent. Conversely,*

(ii) *If $E\mathcal{S}$ and EK are elementarily definitionally equivalent, and if, in addition, $\mathcal{S}$ is protoalgebraic, then $\mathcal{S}$ is algebraizable with equivalent semantics* K.

Proof. The models of $E\mathcal{S}$ are exactly the $\mathcal{S}$-matrices (the matrix models of $\mathcal{S}$). In view of Lemma 5.2, a $\mathcal{S}$-matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$ satisfies the first-order sentence

$$\forall p \forall q (p \approx q \leftrightarrow D(p \,\Delta\, q)) \tag{1}$$

iff $\boldsymbol{\Omega}_{\mathbf{A}} F = I_A$, i.e., iff $\mathcal{A}$ is reduced. Let E be the extension of $E\mathcal{S}$ over the language $\mathcal{L}_{D,\approx}$ axiomatized by (1). Then the models of E are exactly the reduced $\mathcal{S}$-matrices. Thus, since every $\mathcal{S}$-matrix $\langle \mathbf{A}, F\rangle$ is elementarily equivalent (as an $\mathcal{L}_D$-structure) to a reduced matrix (the quotient matrix $\langle \mathbf{A}/\boldsymbol{\Omega}_{\mathbf{A}} F,\ F/\boldsymbol{\Omega}_{\mathbf{A}} F\rangle$), it follows that E is a conservative extension of $E\mathcal{S}$.

Now assume that $\mathcal{S}$ is algebraizable, and let K be its equivalent quasivariety semantics. Let EK be the elementary theory of K over the language $\mathcal{L}_{\approx}$. Since $\mathbf{A}$ is a model of EK (i.e., a member of K) iff $\mathbf{A}$ is the reduct of a reduced $\mathcal{S}$-matrix (by 5.3), we see that E is also a conservative extension of EK. So we have that $E\mathcal{S}$ and EK are elementarily definitionally equivalent by means of the possible definitions (1) and

$$\forall p (Dp \leftrightarrow \delta(p) \approx \epsilon(p))$$

where $\delta(p) \approx \epsilon(p)$ is any set of defining equations for K. (The above argument shows that this latter definition is actually a logical consequence of $E\mathcal{S}$ together with (1).) This proves A.1(i).

To prove part (ii) of A.1 we assume that $E\mathcal{S}$ and EK are elementarily definitionally equivalent with possible definitions

$$\forall p \forall q (p \approx q \leftrightarrow \alpha(p, q)), \tag{2}$$

$$\forall p (Dp \leftrightarrow \beta(p)) \tag{3}$$

where α and β are arbitrary first-order formulas in the languages $\mathcal{L}_D$ and $\mathcal{L}_{\approx}$, respectively. We will use Theorem 3.7 to show that K is an equivalent algebraic semantics for $\mathcal{S}$.

Let E be the $\mathcal{L}_{D,\approx}$-theory axiomatized $E\mathcal{S} \cup E$K together with the possible definitions (2) and (3). For each $T \in Th\mathcal{S}$ let $\mathcal{F}_T = \langle \mathbf{Fm}_{\mathcal{L}}, T\rangle$ be the formula matrix associated with T, and let $\Omega_\alpha T$ be the binary relation on Fm defined by

$$\Omega_\alpha T = \{\langle \xi, \eta\rangle : \mathcal{F}_T \models \alpha[\xi, \eta]\}\,.$$

We show that Ω_α is a isomorphism between $\mathbf{Th}\mathcal{S}$ and $\mathbf{Th}\mathsf{K}$ that commutes with surjective substitutions.

For each first-order sentence γ of $\mathcal{L}_{D,\approx}$, let $\tilde{\gamma}$ be the equality-free sentence in $\mathcal{L}_D$ obtained from γ by replacing every atomic subformula of the form $\xi \approx \eta$ by $\alpha(\xi, \eta)$. Then we have

$$E \vdash \gamma \Leftrightarrow ES \vdash \tilde{\gamma}. \tag{4}$$

For suppose $E \vdash \gamma$, i.e, $\gamma \in E$. Then definition (2) implies that $\tilde{\gamma} \in E$. But $\tilde{\gamma}$ is a sentence of $\mathcal{L}_D$, and since E is a conservative extension of ES we must have $\tilde{\gamma} \in ES$. The implication in the opposite direction is obvious.

(4) implies that $\Omega_\alpha T$ has all the properties of the identity relation that can be formulated in the language $\mathcal{L}_D$. This is equivalent to saying that it is a congruence on $\mathbf{Fm}$ (i.e., an equational theory in the sense of Chapter 2), and that $\Omega_\alpha T$ is compatible with T. So by 1.6(ii) $\Omega_\alpha T = \mathbf{\Omega} T$, the Leibniz equality relation on $\mathbf{Fm}$ associated with the theory T.

Define

$$H_\beta \Theta = \{\varphi \in Fm_{\mathcal{L}} : \mathbf{Fm}_{\mathcal{L}}/\Theta \models \beta[\varphi]\}$$

for each $\Theta \in Th\,\mathsf{K}$. Observe that for each $T \in Th\mathcal{S}$ we have $\varphi \in H_\beta \mathbf{\Omega} T$ iff $\mathbf{Fm}_{\mathcal{L}}/\mathbf{\Omega} T \models \beta[\varphi]$ iff $\mathcal{F}_T \models \tilde{\beta}[\varphi]$ (by definition of $\Omega_\alpha T$). But taking γ in (4) to be the possible definition (3) we get $\mathcal{F}_T \models \tilde{\beta}[\varphi]$ iff $\varphi \in T$. Thus $H_\beta \mathbf{\Omega} T = T$. In a similar manner it can be shown that $\mathbf{\Omega} H_\beta \Theta = \Theta$ for each $\Theta \in Th\,\mathsf{K}$. Hence the Leibniz operator $\mathbf{\Omega}$ is a bijection between $Th\mathcal{S}$ and $Th\,\mathsf{K}$, and, since $\mathcal{S}$ is protoalgebraic by hypothesis, $\mathbf{\Omega}$ is actually an isomorphism between the theory lattices of $\mathcal{S}$ and K. It follows immediately that the hypothesis of Lemma 4.6 holds, and hence that $\mathbf{\Omega}$ commutes with surjective substitutions. We can now apply 3.7 to conclude that $\mathcal{S}$ is algebraizable with equivalent semantics K. ∎

In defining the notion of equivalent algebraic semantics, and by means of it the concept of an algebraizable logic, we required that the language of K and that of $\mathcal{S}$ be the same. This is a natural course to take in view of the historical development of algebraic logic, but in the present context it appears somewhat arbitrary. We may drop this requirement and allow possible definitions for the primitive connectives of $\mathcal{S}$, as well as for the predicate D, and similarly for the operation symbols of K. Theorem A.1, in an appropriately modified form, would continue to hold. We would however get a more general notion of equivalent algebraic semantics (but not of algebraizability). A quasivariety K would be an equivalent algebraic semantics for $\mathcal{S}$ in this more general sense iff it is an equivalent algebraic semantics in the original sense for another deductive system $\mathcal{S}'$ that formalizes the same logic as $\mathcal{S}$ on the basis of a different set of primitive connectives.

Appendix B

An Example

The Leibniz operator $\mathbf{\Omega}$ is injective and order-preserving on the lattice of theories of every algebraizable deductive system. Conversely, if $\mathbf{\Omega}$ has these properties, then the system is algebraizable, provided $\mathbf{\Omega}$ also preserves the join of each directed family of theories (Theorem 4.2). We had conjectured that the property of being order-preserving and injective is enough to guarantee algebraizability. H. Andréka and I. Németi have constructed a counterexample to this conjecture however, which is also interesting for other reasons. We present a slightly modified version here with their kind permission.

Let $\mathcal{L} = \{\Delta, \star\}$ with Δ a binary and $\star$ a unary connective. Let $\mathcal{S}$ be the deductive system over $\mathcal{L}$ defined by the axioms

$$\star p, \quad p \,\Delta\, p,$$

and the infinite family of inference rules

$$p, p \,\Delta\, p \vdash_{\mathcal{S}} q, \qquad (detachment)$$

$$p \vdash_{\mathcal{S}} \vartheta \,\Delta\, \vartheta[\star p/p], \quad \text{for each } \vartheta \in Fm_{\mathcal{L}},$$

$$p \vdash_{\mathcal{S}} \vartheta[\star p/p] \,\Delta\, \vartheta, \quad \text{for each } \vartheta \in Fm_{\mathcal{L}}.$$

Theorem B.1 $\mathbf{\Omega}$ *is injective and order-preserving on* $\mathbf{Th}\mathcal{S}$*, but it does not preserve unions of directed subsets of* $Th\mathcal{S}$*. Hence* $\mathcal{L}$ *is not algebraizable.*

The proof will be given in a sequence of lemmas.

Lemma B.2 $\mathbf{\Omega}$ *is injective and order-preserving.*

Proof. We first show $\mathbf{\Omega}$ is order-preserving. Suppose S and T are $\mathcal{S}$-theories such that $S \subseteq T$. We must show $\mathbf{\Omega}S \subset \mathbf{\Omega}T$. $\mathbf{\Omega}S$ is a congruence on $\mathbf{Fm}_{\mathcal{L}}$. So it suffices to show that $\mathbf{\Omega}S$ is compatible with T (Theorem 2.7). Suppose $\langle\varphi, \psi\rangle \in \mathbf{\Omega}S$ and $\varphi \in T$. Then $\langle \varphi \,\Delta\, \varphi, \varphi \,\Delta\, \psi\rangle \in \mathbf{\Omega}S$ since $\mathbf{\Omega}S$ is a congruence. But $\varphi \,\Delta\, \varphi \in S$ by the second axiom. Thus $\varphi \,\Delta\, \psi \in S \subseteq T$ because $\mathbf{\Omega}S$ is compatible with S. Hence we conclude finally that $\psi \in T$ by detachment.

To show $\mathbf{\Omega}$ is injective, it clearly suffices to show that, for each $T \in Th\mathcal{S}$,

$$\varphi \in T \Longleftrightarrow \langle \varphi, \star\varphi \rangle \in \mathbf{\Omega}T. \tag{1}$$

Suppose $\varphi \in T$. Then by the last two inference rules we have

$$\vartheta[\varphi/p] \mathbin{\Delta} \vartheta[\star\varphi/p],\ \vartheta[\star\varphi/p] \mathbin{\Delta} \vartheta[\varphi/p] \in T$$

for every $\vartheta \in Fm$. Thus, by detachment, $\vartheta[\varphi/p] \in T$ iff $\vartheta[\star\varphi/p] \in T$ for every $\vartheta \in Fm$, and so $\langle \varphi, \star\varphi \rangle \in \mathbf{\Omega}T$. Suppose, conversely, that $\langle \varphi, \star\varphi \rangle \in \mathbf{\Omega}T$. Then $\langle \varphi \mathbin{\Delta} \varphi, \star\varphi \mathbin{\Delta} \varphi \rangle \in \mathbf{\Omega}T$. Since $\varphi \mathbin{\Delta} \varphi \in T$, and $\mathbf{\Omega}T$ is compatible with T, $\star\varphi \mathbin{\Delta} \varphi \in T$. But $\star\varphi \in T$ by the first axiom. Hence $\varphi \in T$ by detachment. This establishes (1), and thus $\mathbf{\Omega}$ is injective in $\mathbf{Th}\mathcal{S}$. ∎

The difficult part of the proof of B.1 is establishing that $\mathbf{\Omega}$ does not preserve unions of directed sets of theories. For this purpose we construct an infinite chain $T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$ of $\mathcal{S}$-theories with the property $\bigcup_{n<\omega} \mathbf{\Omega}(T_n) \neq \mathbf{\Omega}(\bigcup_{n<\omega} T_n)$. We first define some useful auxiliary notions.

Let $n < \omega$ be fixed but arbitrary; the length of a formula φ will be denoted by $|\varphi|$. Define

$$S_n = \{\star\varphi : \varphi \in Fm\} \cup \{\varphi \mathbin{\Delta} \varphi : \varphi \in Fm\} \cup \{\varphi : |\varphi| \leq n\}.$$

S_n is not a theory, but will eventually be transformed into one. For this purpose we define a binary reduction relation $\Rightarrow_n$ on formulas by the condition that $\varphi \Rightarrow_n \psi$ iff φ contains a subterm of the form $\star\xi$ with $\xi \in S_n$, and ψ is obtained from φ by replacing $\star\xi$ by ξ, i.e., by deleting the $\star$. In symbols,

$$\varphi \Rightarrow_n \psi \quad \text{iff} \quad \varphi = \vartheta[\star\xi/p] \text{ and } \psi = \vartheta[\xi/p] \text{ for some } \vartheta \in Fm \text{ and } \xi \in S_n.$$

Let $\Rightarrow_n^*$ be the reflexive, transitive closure of $\Rightarrow_n$. $\Rightarrow_n^*$ is *well founded* in the strong sense that, for each formula φ, there exists an l such that $\varphi \Rightarrow_n \varphi_1 \Rightarrow_n \varphi_2 \Rightarrow_n \dots \Rightarrow_n \varphi_m$ implies $m \leq l$.

In the following two lemmas we establish the basic properties of $\Rightarrow_n$ and $\Rightarrow_n^*$ that we shall need.

Lemma B.3 (i) *If $\varphi \Rightarrow_n^* \psi$, then $\vartheta[\varphi/p] \Rightarrow_n \vartheta[\psi/p]$ for every ϑ and p.*

(ii) *$\varphi \mathbin{\Delta} \psi \Rightarrow_n \vartheta$ iff $\vartheta = \varphi' \mathbin{\Delta} \psi'$ with either $\varphi \Rightarrow_n \varphi'$ and $\psi = \psi'$, or $\varphi = \varphi'$ and $\psi \Rightarrow_n \psi'$.*

(iii) *$\star\varphi \Rightarrow_n \psi$ iff $\psi = \varphi \in S_n$, or $\varphi \Rightarrow_n \vartheta$ with $\star\vartheta = \psi$.*

(iv) *If $\varphi \in S_n$ and $\varphi \Rightarrow_n \psi$, then $\psi \Rightarrow_n^* \vartheta$ for some $\vartheta \in S_n$.*

Proof. (i)–(iii) are obvious. To show (iv), suppose first of all the $\varphi = \star\varphi'$. Then either $\psi = \varphi' \in S_n$, or $\varphi' \Rightarrow_n \vartheta$ with $\star\vartheta = \psi$; $\psi \in S_n$ in both cases. Suppose next that $\varphi = \varphi' \,\Delta\, \varphi'$. Then $\varphi' \Rightarrow_n \vartheta$, and $\psi = \vartheta \,\Delta\, \varphi'$ or $\psi = \varphi' \,\Delta\, \vartheta$. Thus $\psi \Rightarrow_n \vartheta \,\Delta\, \vartheta \in S_n$. Finally, the result is obvious if $|\varphi| \leq n$. ∎

The next lemma establishes the so-called *Church-Rosser property* for $\Rightarrow_n^*$.

Lemma B.4 *If $\varphi \Rightarrow_n^* \psi$ and $\varphi \Rightarrow_n^* \psi'$, then there exists a κ such that $\psi \Rightarrow_n^* \kappa$ and $\psi \Rightarrow_n^* \kappa$.*

Proof. Because $\Rightarrow_n^*$ is well-founded in the strong sense, we lose no generality by assuming $\varphi \Rightarrow_n \psi$ and $\varphi \Rightarrow_n \psi'$. Thus $\varphi = \vartheta[\star\xi/p]$ and $\varphi = \vartheta'[\star\xi'/p']$, with $\xi, \xi' \in S_n$ and $\psi = \vartheta[\xi/p]$ and $\psi' = \vartheta'[\xi'/p']$. Suppose the occurrences of $\star\xi$ and $\star\xi'$ in ϑ are disjoint. Then $\vartheta = \lambda[\star\xi/p, \star\xi'/p']$ with $\psi = \lambda[\xi/p, \star\xi'/p']$ and $\psi' = \lambda[\star\xi/p, \xi'/p']$. We can take $\kappa = \lambda[\xi/p, \xi'/p']$.

Suppose one of $\star\xi$ or $\star\xi'$ occurs in the other, say $\xi = \lambda[\star\xi'/p']$. Then $\varphi = \vartheta[\star\lambda[\star\xi'/p']/p]$ and $\psi = \vartheta[\lambda[\star\xi'/p']/p]$ and $\psi' = \vartheta[\star\lambda[\xi'/p']/p]$. Observe that, since $\lambda[\star\xi'/p'] = \xi \in S_n$, and $\lambda[\star\xi'/p'] \Rightarrow_n \lambda[\xi'/p']$, we have $\lambda[\xi'/p'] \Rightarrow_n^* \eta \in S_n$ by Lemma B.2(iv). Take $\kappa = \vartheta[\eta/p]$. Then $\psi = \vartheta[\lambda[\star\xi'/p']/p] \Rightarrow_n \vartheta[\lambda[\xi'/p']/p] \Rightarrow_n^* \vartheta[\eta/p] = \kappa$, and $\psi' = \vartheta[\star\lambda[\xi'/p']/p] \Rightarrow_n^* \vartheta[\star\eta/p] \Rightarrow_n \vartheta[\eta/p] = \kappa$. ∎

We denote by $\Rightarrow_n^*\Leftarrow$ the binary relation $(\Rightarrow_n^*) \mid (\Rightarrow_n^*)^{-1}$ on Fm (i.e., the relative product of $\Rightarrow_n^*$ with its converse). So $\varphi \Rightarrow_n^*\Leftarrow \psi$ iff there exists a $\vartheta \in Fm$ such the $\varphi \Rightarrow_n^* \vartheta$ and $\psi \Rightarrow_n^* \vartheta$. We also write ${}_n^*\!\Leftarrow$ for $(\Rightarrow_n)^{-1}$. Note that $\Rightarrow_n$ and ${}_n^*\!\Leftarrow$ are both included in $\Rightarrow_n^*\Leftarrow$.

Lemma B.5 *$\Rightarrow_n^*\Leftarrow$ is an equivalence relation on Fm.*

Proof. $\Rightarrow_n^*\Leftarrow$ is clearly reflexive and symmetric. Assume $\varphi \Rightarrow_n^*\Leftarrow \psi$ and $\psi \Rightarrow_n^*\Leftarrow \vartheta$. Let $\varphi \Rightarrow_n^* \kappa$, $\psi \Rightarrow_n^* \kappa$ and $\psi \Rightarrow_n^* \lambda$, $\vartheta \Rightarrow_n^* \lambda$. By the Church-Rosser property there is a μ such that $\kappa \Rightarrow_n^* \mu$ and $\lambda \Rightarrow_n^* \mu$. Thus $\varphi \Rightarrow_n^* \kappa \Rightarrow_n^* \mu$ and $\vartheta \Rightarrow_n^* \lambda \Rightarrow_n^* \mu$. So $\varphi \Rightarrow_n^*\Leftarrow \vartheta$. Thus $\Rightarrow_n^*\Leftarrow$ is transitive. ∎

Define

$$T_n = \{\, \varphi : \varphi \Rightarrow_n^*\Leftarrow \psi \text{ for some } \psi \in S_n \,\}.$$

Lemma B.6 *T_n is a theory.*

Proof. T_n contains all the axioms since S_n does and $S_n \subseteq T_n$. Suppose φ, $\varphi \,\Delta\, \psi \in T_n$. Then $\varphi \Rightarrow_n^*\Leftarrow \varphi' \in S_n$, and $\varphi \,\Delta\, \psi \Rightarrow_n^*\Leftarrow \varphi'' \,\Delta\, \psi'' \in S_n$ where $\varphi \Rightarrow_n^* \varphi''$ and $\psi \Rightarrow_n^* \psi''$. $\varphi'' \,\Delta\, \psi'' \in S_n$ implies either $\varphi'' = \psi''$ or $|\varphi'' \,\Delta\, \psi''| \leq n$.

In the latter case $|\psi''| \le n$, and hence $\psi'' \in S_n$; so $\psi \in T_n$. We can now assume $\varphi'' = \psi''$. Thus $\varphi \Rightarrow_n^* \Leftarrow \varphi' \in S_n$. $\varphi \Rightarrow_n^* \Leftarrow \psi''$, and $\psi \Rightarrow_n^* \Leftarrow \psi''$. Since $\Rightarrow_n^* \Leftarrow$ is an equivalence relation, $\psi \Rightarrow_n^* \Leftarrow \varphi' \in S_n$, and hence $\psi \in T_n$.

Now assume $\varphi \in T_n$. Then $\varphi \Rightarrow_n^* \Leftarrow \varphi' \in S_n$. Let $\varphi \Rightarrow_n^* \psi$ and $\varphi' \Rightarrow_n^* \psi$. Then $\vartheta[\varphi/p] \,\Delta\, \vartheta[\star\varphi/p] \Rightarrow_n^* \vartheta[\varphi/p] \,\Delta\, \vartheta[\star\psi/p] {}_n^*\!\Leftarrow \vartheta[\varphi/p] \,\Delta\, \vartheta[\star\varphi'/p] \Rightarrow_n \vartheta[\varphi/p] \,\Delta\, \vartheta[\varphi'/p] \Rightarrow_n^* \vartheta[\varphi/p] \,\Delta\, \vartheta[\psi/p] {}_n^*\!\Leftarrow \vartheta[\varphi/p] \,\Delta\, \vartheta[\varphi/p]$. So

$$\vartheta[\varphi/p] \,\Delta\, \vartheta[\star\varphi/p] \Rightarrow_n^* \Leftarrow \vartheta[\varphi/p] \,\Delta\, \vartheta[\varphi/p];$$

hence $\vartheta[\varphi/p] \,\Delta\, \vartheta[\star\varphi/p] \in T_n$. Similarly, $\varphi \in T_n$ implies $\vartheta[\star\varphi/p] \,\Delta\, \vartheta[\varphi/p] \in T_n$. ∎

Lemma B.7 *If ϑ contains no occurrence of $\star$ and $\varphi \in T_n$, then $\varphi \in S_n$.*

Proof. Assume $\varphi \Rightarrow_n^* \Leftarrow \psi \in S_n$, say $\varphi \Rightarrow_n^* \vartheta$ and $\psi \Rightarrow_n^* \vartheta$. Because of the hypothesis on φ, we must have $\varphi = \vartheta$. Thus $\psi \Rightarrow_n^* \varphi$ and $\psi \in S_n$. Let ψ be a formula of minimal length satisfying this condition. Suppose $\psi \neq \varphi$. Then $\psi \Rightarrow_n \psi' \Rightarrow_n^* \varphi$. By Lemma B.3(iv), $\psi' \Rightarrow_n^* \psi''$ for some $\psi'' \in S_n$. By the Church-Rosser property, $\psi'' \Rightarrow_n^* \Leftarrow \varphi$, and by hypothesis on φ, $\psi'' \Rightarrow_n^* \varphi$. But $|\psi''| < |\psi|$, contradicting the assumption. So $\varphi = \psi \in S_n$. ∎

Proof of Theorem B.1: $\mathbf{\Omega}$ is injective and order-preserving by Lemma B.2. So it only remains to show that it does not preserve joins of directed sets. Observe that $\bigcup_{n<\omega} T_n \supseteq \bigcup_{n<\omega} S_n = Fm$, so $\mathbf{\Omega}(\bigcup_{n<\omega} T_n)$ is the universal relation on Fm. Let p and q be distinct variables. $(\Delta^n p^{n+1}) \,\Delta\, (\Delta^n q^{n+1}) \notin S_n$, and hence it is not in T_n by the last lemma. Thus $\langle p, q\rangle \notin \mathbf{\Omega} T_n$ for all n. ∎

The deductive system S of Theorem B.1 was defined by an infinite number of inference rules. The conjecture mentioned at the beginning of the appendix might still be true if we consider only finite axiomatizable deductive systems; the problem is open. It is also not known if the conjecture holds for the so-called *strongly finite* systems; these are the deductive systems defined by a finite set of finite matrices.

Appendix C

Predicate Logic

The problem of algebraizing predicate logic is of a different character than the problem for propositional logics because the standard deductive systems for predicate logic are not structural. They fail to be structural for two closely related reasons. The first has to do with the ambiguity inherent in the use of individual variables for two essentially different purposes—as both free and bound variables. The second is connected with the way the systems deal with the process of substituting terms for the free occurrences of an individual variable in a formula. Consider for example the $\forall$-elimination and $\forall$-introduction rules of Kleene [23]:

$$\forall v\varphi(v) \to \varphi(t) \quad \text{and} \quad \psi \to \varphi(v) \vdash \psi \to \forall v\varphi(v).$$

In the first t is a term free for v in $\varphi(v)$, and $\varphi(t)$ is the result of substituting t for all free occurrences of v. In the second v does not occur free in ψ. Neither of the rules is structural. (The first is actually an axiom; an axiom is ***structural*** if all its substitution instances are axioms.) $\varphi(v)$ and $\varphi(t)$ cannot be treated as distinct formula variables; similarly, ψ cannot be considered a variable because of the stipulation that it contains no free occurrence of v. The problem is that the standard deductive systems for predicate logic all deal with substitution, and the distinction between free and bound variables, informally in the metalanguage; to obtain a structural system these things have to be handled formally within the system itself. Such systems do exist and they are algebraizable.

There are different ways of treating substitution formally, and these lead to deductive systems with essentially different equivalent quasivariety semantics. Two main ways of doing this have been developed in the literature. The formula $\varphi(t)$ considered above in connection with the $\forall$-elimination rule is logically equivalent to $\exists v(v = t \wedge \varphi)$, provided v does not occur in t. So $\forall$-elimination can possibly be replaced by the (almost) structural rule $\forall v\varphi(v) \to \exists v(v = t \wedge \varphi)$. The algebraization of deductive systems that exploit this fact give rise to the variety of *cylindric algebras*. One can also deal with substitution more directly by introducing into the formal language a substitution operator S_σ for each mapping σ of the set of individual variables into the set of terms.

In this case $\varphi(t)$ is identified with $S_{[t/v]}\varphi$. The class of algebras that arise in this way are called *polyadic algebras*. Here we only discuss the process that leads to cylindric algebras; the formal development of the one that leads to polyadic algebras is quite similar.

In this context it is customary to deal only with first-order logics that contain no extra-logical operation symbols. This restriction, although not essential, greatly simplifies the associated structural deductive system, and it is well known that every first-order logic is equivalent in a natural sense to a logic with this property. We will describe how to massage the standard first-order language into a propositional language of the type we have been dealing with in this paper. The propositional language we will actually define corresponds to a first-order language that differs in several important respects from the standard first-order language that is normally considered. First of all, it can have any finite or infinite number of individual variables; we assume they are canonically ordered in a sequence $v_0, v_1, \dots, v_\xi, \dots$, for $\xi < \alpha$, where α is a fixed but arbitrary ordinal. α is called the *dimension* of the language. There is a denumerable set $\mathcal{P} = \{P_0, P_1, \dots\}$ of relation symbols all of which are assumed to be of rank α. (So each P_n is infinitary whenever α is infinite.) Standard first-order languages all of whose relation symbols are *generic* in this sense—i.e., their ranks coincide with the dimension of the language—are called *full* languages in Henkin, Monk, and Tarski [15, Part II]. Finally, all atomic formulas, which do not involve equality, are of the form $P_n v_0 v_1 \dots v_\xi \dots$, with the variable arguments occurring in canonical order; languages with this property are called *restricted* in [15, Part II].

This language will be called the (*restricted, **full***) ***first-order language of*** *dimension* α (*over* $\mathcal{P}$). The propositional language associated with it is denoted by $\mathcal{L}_\alpha$. Besides the usual sentential connectives $\vee$, $\wedge$, $\neg$, $\to$, $\top$, and $\bot$, it has two α-sequences of primitive unary connectives

$$\forall v_0, \forall v_1, \dots, \forall v_\xi, \dots,\ \exists v_0, \exists v_1, \dots, \exists v_\xi, \dots,$$

for $\xi < \alpha$, corresponding to universal and existential quantification over each of the individual variables. $\mathcal{L}_\alpha$ also contains a primitive nullary (constant) symbol $v_\kappa \approx v_\lambda$ for each pair of ordinals $\kappa, \lambda < \alpha$. Each of the symbol complexes $\forall v_\kappa$, $\exists v_\kappa$, and $v_\kappa \approx v_\lambda$ is to be considered indivisible in $\mathcal{L}_\alpha$. It would be better from a logical point of view to denote these connectives by something like $\forall_\kappa$, $\exists_\lambda$, and $\approx_{\kappa,\lambda}$, but for practical reasons it seems advisable to keep $\mathcal{L}_\alpha$, in form at least, as close to the first-order language as possible.

$\mathcal{L}_\alpha$, like all the languages we consider, also contains the infinite sequence of propositional variables $\mathbf{p}_0, \mathbf{p}_1, \dots$. It is appropriate in this context to call these *relation variables*, rather than propositional variables. They correspond to the atomic formulas (in restricted form)

$$P_0 v_0 \dots v_\xi \dots,\ P_1 v_0 \dots v_\xi \dots, \dots$$

of the first-order language.

The deductive system $\mathbf{PR}_\alpha$ of first-order predicate logic over the language $\mathcal{L}_\alpha$ is defined by the following axioms and rules of inference; κ, λ, and μ range over all ordinals $< \alpha$, and p, q, and r stand for $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_3$, respectively:

A1 all classical tautologies,
A2 $\forall v_\kappa (p \to q) \to (\forall v_\kappa p \to \forall v_\kappa q)$,
A3 $\forall v_\kappa p \to p$,
A4 $\forall v_\kappa \forall v_\lambda p \to \forall v_\lambda \forall v_\kappa p$,
A5 $\forall v_\kappa p \to \forall v_\kappa \forall v_\kappa p$,
A6 $\exists v_\kappa p \to \forall v_\kappa \exists v_\kappa p$,
A7 $v_\kappa \approx v_\kappa$,
A8 $\exists v_\kappa (v_\kappa \approx v_\lambda)$,
A9 $v_\kappa \approx v_\lambda \to (v_\kappa \approx v_\mu \to v_\lambda \approx v_\mu)$,
A10 $v_\kappa \approx v_\lambda \to (p \to \forall v_\kappa (v_\kappa \approx v_\lambda \to p))$, if $\kappa \neq \lambda$,
A11 $\exists v_\kappa p \leftrightarrow \neg \forall v_\kappa \neg p$,
R1 $p, p \to q \vdash_{\mathbf{PR}_\alpha} q$, *(modus ponens)*
R2 $p \vdash_{\mathbf{PR}_\alpha} \forall v_\kappa p$. *(generalization)*.

This formulation of the predicate calculus is due to I. Németi. A closely related but non-structural one can be found in Monk [31]; see also Henkin, Monk, and Tarski [15, Part II, p.157]. An earlier effort to formulate predicate logic in a propositional language is undertaken in Jaśkowski [18].

Like all deductive systems we consider, $\mathbf{PR}_\alpha$ is structural. Each axiom and inference rule can be considered a schema that includes, along with itself, all its substitution instances. So the relation variables represent relations of all possible ranks. This is why the relation symbols $P_0, P_1, \dots$ in the full first-order language, which correspond to the the relation variables of $\mathcal{L}_\alpha$, must be generic. In a structural deductive system that attempts to model normal, finitary predicate logic the relation symbols cannot be interpreted as variables. New nullary symbols (i.e., constants) must be adjoined to $\mathcal{L}_\alpha$ for this purpose.

Let $\mathcal{R} = \{R_i : i \in I\}$ be a system of constant relation symbols, and let $\rho : I \to \alpha$ be the corresponding rank function. Let $\mathcal{L}_\alpha^\mathcal{R}$ be the language obtained from $\mathcal{L}_\alpha$ by adjoining for each $R_i \in \mathcal{R}$ a single new nullary symbol $R_i v_0 \dots v_\xi \dots$, $\xi < \rho i$, where $v_0 \dots v_\xi \dots$ is an initial segment of the canonical sequence $v_0, v_1, \dots, v_\xi, \dots$ of individual variable symbols. The corresponding first-order language over $\mathcal{P} \cup \mathcal{R}$ is no longer full but remains restricted. We denote by $\mathbf{PR}_\alpha^\mathcal{R}$ the deductive system over $\mathcal{L}_\alpha^\mathcal{R}$ defined by axioms A1–A11, the rules R1, R2, and the additional axiom

A12 $R_i v_i \ldots v_\xi \ldots \to \forall v_\kappa R_i v_0 \ldots v_\xi \ldots,\quad i \in I$ and $\rho i \leq \kappa < \alpha$.

Now let $\mathcal{R}$ be a set of finitary constant relation symbols, and consider the first-order language over $\mathcal{R}$ as it is ordinarily defined in a textbook in logic. We shall call this the *standard first-order language over* $\mathcal{R}$. It is of dimension ω, and contains no relation variables. It is unrestricted in the sense that atomic formulas of the form $R_i v_{\sigma 0} \ldots v_{\sigma(\rho i - 1)}$ may occur where the $v_{\sigma 0}, \ldots, v_{\sigma(\rho i-1)}$ are not in canonical order. By considering only formulas whose atomic subformulas are in restricted form we get the *restricted standard first-order language over* $\mathcal{R}$. By limiting attention to restricted formulas we do not diminish the expressive power of the standard first-order language since every standard formula is logically equivalent to one in restricted form. For example, if R_i is binary, then $R_i v_1 v_0$ is logically equivalent to

$$\exists v_2 \exists v_3 (v_1 \approx v_2 \wedge v_0 \approx v_3 \wedge \exists v_0 \exists v_1 (v_0 \approx v_2 \wedge v_1 \approx v_3 \wedge R_i v_0 v_1)).$$

In [31] Monk proves that the axioms A1–A12 and rules R1, R2 provide a sound and complete formalization of first-order predicate logic[1] in the following sense: Let them be interpreted as formula and rule schemata in the restricted standard language of $\mathcal{R}$. (I.e., the relation variables p, q, and r are to be interpreted as metavariables ranging over all restricted standard formulas.) Then an arbitrary restricted formula is logically valid iff it is derivable from A1–A12 in the usual way by means of the rules R1 and R2.

The following definition is taken from Henkin, Monk, and Tarski [15, Part I, p.162].

By a *cylindric algebra of dimension* α, where α is any ordinal number, we mean an algebraic structure

$$\mathbf{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_\kappa, \mathsf{d}_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha}$$

such that 0, 1, and $\mathsf{d}_{\kappa\lambda}$ are distinguished elements of A, $-$ and c_κ are unary operations on A, $+$ and $\cdot$ are binary operations on A, and such that the following postulates are satisfied for all $x, y \in A$ and all $\kappa, \lambda, \mu < \alpha$:

[1] Monk's axiom system is slightly different from the one given here, but his proof is easily modified.

C0 the structure $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra,
C1 $\mathsf{c}_\kappa 0 = 0$,
C2 $x \leq \mathsf{c}_\kappa x$ (i.e., $x + \mathsf{c}_\kappa x = \mathsf{c}_\kappa x$),
C3 $\mathsf{c}_\kappa(x \cdot \mathsf{c}_\kappa y) = \mathsf{c}_\kappa x \cdot \mathsf{c}_\kappa y$,
C4 $\mathsf{c}_\kappa \mathsf{c}_\lambda x = \mathsf{c}_\lambda \mathsf{c}_\kappa x$,
C5 $\mathsf{d}_{\kappa\kappa} = 1$,
C6 $\mathsf{d}_{\lambda\mu} = \mathsf{c}_\kappa(\mathsf{d}_{\lambda\kappa} \cdot \mathsf{d}_{\kappa\gamma})$, if $\kappa \neq \lambda$,
C7 $\mathsf{c}_\kappa(\mathsf{d}_{\kappa\lambda} \cdot x) \cdot \mathsf{c}_\kappa(\mathsf{d}_{\kappa\lambda} \cdot -x) = 0$, if $\kappa \neq \lambda$.

The class of all cylindric algebras of dimension α is denoted by CA_α. The elements $\mathsf{d}_{\kappa\lambda}$ are called *diagonal elements*, and the operations c_κ are called *cylindrifications*.

Let $\mathcal{R}$ be an arbitrary set of constant relation symbols, and adjoin a new nullary symbol r_i to the language of CA_α for each $R_i \in \mathcal{R}$. If we identify $+$, $\cdot$, $-$, 0, 1, c_κ, $\mathsf{d}_{\kappa\lambda}$, and r_i, respectively with $\vee$, $\wedge$, $\neg$, $\bot$, $\top$, $\exists v_\kappa$, $v_\kappa \approx v_\lambda$, and $R_i v_0 \dots v_\xi \dots$, with $\xi < \rho i$, and if we interpret $x \rightarrow y$ as $-x + y$ and $\forall v_\kappa x$ as $-\mathsf{c}_\kappa - x$, we get an algebra over the language $\mathcal{L}_\alpha^{\mathcal{R}}$. We denote by $\mathsf{CA}_\alpha^{\mathcal{R}}$ the variety of algebras over $\mathcal{L}_\alpha^{\mathcal{R}}$ defined by the identities C1–C7 together with the constant identities

(C8) $\mathsf{c}_\kappa r_i = r_i$, for all $i \in I$ and $\rho i \leq \kappa < \alpha$.

Theorem C.1 *Let α be any ordinal number.*

(i) $\mathbf{PR}_\alpha$ *is algebraizable, and its equivalent quasivariety semantics is definitionally equivalent to* CA_α.

(ii) *For any set $\mathcal{R}$ of constant relation symbols,* $\mathbf{PR}_\alpha^{\mathcal{R}}$ *is algebraizable, and its equivalent quasivariety semantics is definitionally equivalent to* $\mathsf{CA}_\alpha^{\mathcal{R}}$.

Proof. (i) is a special class (ii) with $\mathcal{R} = \emptyset$. To prove $\mathbf{PR}_\alpha^{\mathcal{R}}$ is algebraizable we apply Corollary 4.8. Take $\Delta(p, q) = \{p \rightarrow q, q \rightarrow p\}$. Then the consequence relationships

$$\vdash_{\mathbf{PR}_\alpha^{\mathcal{R}}} \varphi \,\Delta\, \varphi, \qquad \varphi \,\Delta\, \psi \vdash_{\mathbf{PR}_\alpha^{\mathcal{R}}} \psi \,\Delta\, \varphi, \qquad \varphi \,\Delta\, \psi, \psi \,\Delta\, \vartheta \vdash_{\mathbf{PR}_\alpha^{\mathcal{R}}} \varphi \,\Delta\, \vartheta,$$

together with detachment and the G-rule:

$$\varphi, \varphi \,\Delta\, \psi \vdash_{\mathbf{PR}_\alpha^{\mathcal{R}}} \psi, \qquad \varphi, \psi \vdash_{\mathbf{PR}_\alpha^{\mathcal{R}}} \varphi \,\Delta\, \psi,$$

all follow at once from A1 and R1. Similarly for the substitution rule

$$\varphi_0 \,\Delta\, \psi_0, \dots, \varphi_{n-1} \,\Delta\, \psi_{n-1} \vdash_{\mathbf{PR}_\alpha^{\mathcal{R}}} \omega\varphi_0 \dots \varphi_{n-1} \,\Delta\, \omega\psi_0 \dots \psi_{n-1}, \tag{1}$$

with $\omega \in \{\vee, \wedge, \neg, \rightarrow\}$. For $\omega = \forall v_\kappa$, (1) follows from A2 and the rules R1 and R2, and this in turn gives (1) with $\omega = \exists v_\kappa$ using A11. Thus conditions 4.7(i)–(iv) and 4.8(v),(vi) are satisfied; so $\mathbf{PR}_\alpha^{\mathcal{R}}$ is algebraizable.

Recall that the relation variables $\mathbf{p}_0, \mathbf{p}_1, \dots$ of $\mathcal{L}_\alpha^{\mathcal{R}}$ can be interpreted as the infinitary restricted atomic formulas

$$P_0 v_0, \dots v_\xi \dots, \ P_1 v_0 \dots v_\xi \dots, \dots.$$

Under this interpretation the one-one correspondence between the constant formulas of $\mathcal{L}_\alpha^{\mathcal{R}}$ and the restricted formulas of the standard first-order language over $\mathcal{R}$, discussed previously, can be extended to a correspondence between the whole of $\mathcal{L}_\alpha^{\mathcal{R}}$ and the entire restricted first-order language over $\mathcal{P} \cup \mathcal{R}$. If this is done then, a straightforward modification of the proof of a result in [15, Part II, p.161, Theorem 4.3.28(ii)] can be used to show that $\mathsf{CA}_\alpha^{\mathcal{R}}$ is definitionally equivalent to

$$\{\mathbf{Fm}_{\mathcal{L}_\alpha^{\mathcal{R}}}/\Omega_\Delta T \ : T \in Th(\mathbf{PR}_\alpha^{\mathcal{R}})\}^{\mathrm{Q}}.$$

Thus $\mathsf{CA}_\alpha^{\mathcal{R}}$ is the equivalent quasivariety semantics for $\mathbf{PR}_\alpha^{\mathcal{R}}$ by Theorem 4.10. ∎

One can also define for each ordinal α a deductive system $\mathbf{L}_\alpha$ over the language $\mathcal{L}_\alpha$ by semantical rather than syntactical means. Consider any $\varphi \in Fm_{\mathcal{L}_\alpha}$, and let $\widehat{\varphi}$ be the corresponding formula in the restricted full first-order language of dimension α over $\mathcal{P}$. Any relational structure $\mathbf{A} = \langle A, P_n^{\mathbf{A}} \rangle_{n<\omega}$ such that $P_n^{\mathbf{A}} \subseteq A^\alpha$ for all $n < \omega$ is called an *α-structure*. The notion of $\widehat{\varphi}$ being *universally satisfied* in $\mathbf{A}$, in symbols $\mathbf{A} \models \widehat{\varphi}$, is defined in the standard way. For all $\varphi_0, \dots, \varphi_{n-1}, \psi \in Fm_{\mathcal{L}_\alpha^{\mathcal{R}}}$ we define $\varphi_0, \dots, \varphi_{n-1} \vdash_{\mathbf{L}_\alpha} \psi$ if, for every α-structure $\mathbf{A}$,

$$\mathbf{A} \models \widehat{\varphi} \quad \text{for all } i < n \quad \Rightarrow \quad \mathbf{A} \models \widehat{\psi}$$

Finally, for any $\Gamma \cup \{\psi\} \subseteq Fm_{\mathcal{L}_\alpha}$, define $\Gamma \vdash_{\mathbf{L}_\alpha} \psi$ if $\Gamma' \vdash_{\mathbf{L}_n} \psi$ for some finite $\Gamma' \subseteq \Gamma$. The deductive system $\mathbf{L}_\alpha^{\mathcal{R}} = \langle \mathcal{L}_\alpha^{\mathcal{R}}, \vdash_{\mathbf{L}_\alpha^{\mathcal{R}}} \rangle$ for any set of non-generic relation symbols is defined similarly.

The theorem of Monk [31] discussed previously can now be reformulated as follows: for each constant formula φ of $\mathcal{L}_\alpha^{\mathcal{R}}$,

$$\vdash_{\mathbf{L}_\omega^{\mathcal{R}}} \varphi \quad \Leftrightarrow \quad \vdash_{\mathbf{PR}_\omega^{\mathcal{R}}} \varphi.$$

In general $\vdash_{\mathbf{L}_\alpha^{\mathcal{R}}}$ and $\vdash_{\mathbf{PR}_\alpha^{\mathcal{R}}}$ do not coincide. But $\mathbf{L}_\alpha^{\mathcal{R}}$ is an extension of $\mathbf{PR}_\alpha^{\mathcal{R}}$, and hence it is also algebraizable (Corollary 4.9). The equivalent quasivariety semantics for $\mathbf{L}_\alpha$ is the class of *generalized cylindric set algebras of dimension* α; see [15, Part II]. Monk [32] proves that $\mathbf{L}_\alpha$ cannot be defined by a finite set of axioms and rules of inference. (He actually deals with the polyadic analogue of $\mathbf{L}_\alpha$.) For a discussion of $\mathbf{L}_\alpha$ for $\alpha \geq \omega$ see Andréka, Gergely, and Németi [3]. A survey of the work that has been done on $\mathbf{L}_\alpha^{\mathcal{R}}$, $\mathbf{PR}_\alpha^{\mathcal{R}}$, and related logics, together with a comprehensive list of references, can be found in [15].

Bibliography

[1] Ackerman, W. *Begründung einer strengen Implikation*. J. Symbolic Logic, **21**(1956), 113–118.

[2] Anderson, A. R. and Belnap, N.D. Jr. *Entailment. The logic of relevance and necessity*. Princeton University Press, Princeton, 1975.

[3] Andréka, H., Gergely, T., and Németi, I. *On universal algebraic constructions of logics*. Studia Logica, **36**(1977), 9–47.

[4] Andréka, H., Németi, I., and Sain, I. *Abstract model theoretic approach to algebraic logic*. Manuscript.

[5] Balbes, R. and Dwinger, P. *Distributive Lattices*. University of Missouri Press, Columbia, Missouri, 1974.

[6] Blok, W. J. and Köhler, P. *Algebraic semantics for quasi-classical modal logics*. J. Symbolic Logic, **48**(1983), 941–964.

[7] Blok, W. J. and Pigozzi, D. *Deduction theorems in algebraic logic*. To appear.

[8] Blok, W. J. and Pigozzi, D. *Protoalgebraic logics*. Studia Logica, **45** (1986), 337–369.

[9] Bloom, S. L. *Some theorems on structural consequence relations*. Studia Logica, **34** (1975), 1–9.

[10] Church, A. *The weak theory of implication*. Appears in: *Kontrolliertes Denken, Untersuchungen zum Logikkalkül und der Logik der Einzelwissenschaften*. Ed. Menne-Wilhelmy-Angsil, Kommissions-Verlag Karl Aber, Munich, 1951, pp. 22–37.

[11] Czelakowski, J. *Equivalential logics (I), (II)*. Studia Logica, **40**(1981), 227–236, 335–372.

[12] Diego, A. *Sur les algèbres de Hilbert*. Gauthier-Villars, Paris, 1966.

[13] Felscher, W. and Schulte Mönting, J. *Algebraic and Deductive Consequence Operations*. Appears in: *Universal Algebra and its Links with Logic, Algebra, Combinatorics, and Computer Science. Proc. '25. Arbeitstagung über algemeine Algebra', Darmstadt 1983.* Research and Exposition in Mathematics, vol. 4, Eds. P. Burmeister et al., Heldermann Verlag, Berlin, 1984.

[14] Grätzer, G. *Universal Algebra*. Second edition. Springer-Verlag, New York, 1979.

[15] Henkin, L, Monk, J. D., and Tarski, A. *Cylindric algebras, Parts I and II*. North-Holland Publishing Co., Amsterdam, 1971, 1985.

[16] Iséki, K. *An algebra related with a propositional calculus*. Proc. Japan Acad., **42**(1966), 26–29.

[17] Iséki, K. and Tanaka, S. *An introduction to the theory of BCK-algebras*. Math. Japonica, **23**(1978), 1–26.

[18] Jaśkowski, S. *Sur les variables propositionelles dépendentes*. Studia Societatis Scientiarum Torunensis, Sectio A, vol. 1 no. 2, 17–21. Review in Journal of Symbolic Logic **14**(1949), 65.

[19] Kabziński, J. K. *BCI-algebras from the point of view of logic*. Bull. Sect. Logic, Polish Acad. Sci., Inst. Philos. and Socio., **12**(1983), 126–129.

[20] Kabziński, J. K. *Quasivarieties for BCK-logic*. Bull. Sect. Logic, Polish Acad. Sci., Inst. Philos. and Socio., **12**(1983), 130–133.

[21] Kabziński, J. K. *Investigations into the Equivalence Connective*. Rozprawy habilitacyjne no. 48, Uniwersytet Jagielloński, Kraków, 1980.

[22] Kabziński, J. K. and Wroński, A. *On Equivalential Algebras*. Proceedings of the 1975 International Symposium on Multiple-Valued Logic, Indiana University, Bloomington, May 13–16, 1975, 419–428.

[23] Kleene, S. C. *Introduction to metamathematics*. North-Holland Publishing Co., 1971.

[24] Lakser, H. *The structure of pseudocomplemented distributive lattices*. Trans. Amer. Math. Soc., **156**(1971), 335–342.

[25] Łoś, J. and Suszko, R. *Remarks on sentential logics*. Proc. Kon. Nederl. Akad. van Wetenschappen, Series A, **61**(1958), 177–183.

[26] Łukasiewicz, J. *The equivalential calculus.* Appears in: *Jan Łukasiewicz. Selected Works.* Ed. L. Bordowski, North-Holland Publishing Co., Amsterdam, 1970.

[27] Mal′cev, A. I. *On the general theory of algebraic systems (Russian).* Mat. Sb.(N.S.), **35**(**77**)(1954), 3–20.

[28] Mal′cev, A. I. *Algebraic Systems.* Springer-Verlag, New York.

[29] Meredith, C. A. and Prior, A.N., *Investigations into Implicational S5.* Zeitschrift für mathematische Logik und Grundlagen der Mathematik **10**(1964), 203–220.

[30] Meyer, R. K. R_I – *the bounds of finitude.* Zeitschr. für Math. Logik und Grundlagen der Math., **16**(1970), 385–387.

[31] Monk, J. D. *Substitutionless predicate logic with identity.* Arch. Math. Logik Grundlagenforsch., **7**(1965), 102–121.

[32] Monk, J. D. *Provability with finitely many variables.* Proc. Amer. Math. Soc., **27**(1971), 353–358.

[33] Porte, J. *Axiomatization and independence in S4 and S5.* Reports on Mathematical Logic, **16**(1983), 23–33.

[34] Prior, A. N. *Formal Logic.* Second Edition. Clarendon Press, Oxford, 1962.

[35] Prucnal, T. and Wroński, A. *An algebraic characterization of the notion of structural completeness.* Bull. Sect. Logic, Polish Acad. Sci., Inst. Philos. and Socio., **3**(1974), 30–33.

[36] Rasiowa, H. *An algebraic approach to non-classical logics.* North-Holland Publishing Co., Amsterdam, 1974.

[37] Rasiowa, H. and Sikorski, R. *The mathematics of metamathematics.* Państwowe Wydawnictwo Naukowe, Warszawa, 1963.

[38] Rautenberg, W. *Klassische und nicht-klassische Aussagenlogik.* Vieweg Verlag, 1979.

[39] Segerberg, K. *An essay in classical modal logic.* Philosophical Studies, Uppsala, 1971.

[40] Suszko, R. *Identity connective and modality.* Studia Logica, **27**(1971), 7–39.

[41] Suszko, R. *Abolition of the Fregean axiom.* Appears in: *Logic colloquium. Symposium on logic held in Boston, 1971-72.* Ed. R. Parikh, Lecture Notes in Mathematics, Vol. 453, Springer-Verlag, 1975, pp. 169–239.

[42] Tarski, A. *Über einige fundamentale Begriffe der Metamathematik.* Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, **23**(1930), 22–29. English translation appears in [44].

[43] Tarski, A. *Grundzüge des Systemenkalküls. Erster Teil.* Fund. Math., **25**(1935), 503–526. English translation appears in [44].

[44] Tarski, A. *Logic, Semantics, and Metamathematics, papers from 1923 to 1938.* Trans. J. H. Woodger, 2nd Ed., Ed. J. Corcoran, Hackett Pub. Co., Indianapolis, Indiana, 1983.

[45] Tarski, A., Mostowski, A., and Robinson, R. M. *Undecidable Theories.* Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1953.

[46] Tax, R. E. *On the Intuitionistic Equivalential Calculus.* Notre Dame Journal of Formal Logic **14**(1973), 448–456.

[47] Wójcicki, R. *Lectures on propositional calculi.* The Polish Academy of Sciences Institute of Philosophy and Sociology, Ossolineum, The Publishing House of the Polish Academy of Sciences, Warsaw, 1984.

[48] Wójcicki, R. *The theory of propositional calculi. An introduction.* To appear.

[49] Wroński, A. *BCK-algebras do not form a variety.* Math. Japonica, **28**(1983), 211–213.

DEPARTMENT OF MATHEMATICS, STATISTICS
AND COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS AT CHICAGO
CHICAGO, ILLINOIS 60680

DEPARTMENT OF MATHEMATICS
IOWA STATE UNIVERSITY
AMES, IOWA 50011

Index