Hyperbolic Geometry in the High School Geometry Classroom

Christi Donald

Iowa State University

Chapter 1 – Introduction

What is Hyperbolic Geometry? Why should high school students study Hyperbolic Geometry? What role does technology play in the study of Hyperbolic Geometry? This paper will address these questions. In chapter 1, Hyperbolic Geometry will be described as will support for mathematics education regarding the subject. Chapter 2 will focus on the similarities and differences between Euclidean Geometry and Hyperbolic Geometry. A study conducted on teaching Hyperbolic Geometry to high school geometry students will be discussed in Chapter 3.

"Hyperbolic geometry is, by definition, the geometry you get by assuming all the axioms for neutral geometry and replacing Hilbert's parallel postulate by its negation, which we shall call the 'hyperbolic axiom'"(Greenberg, 1993, p. 187). A look at the history of Hyperbolic Geometry will help provide understanding of the definition. Euclidean Geometry gives the foundation for Hyperbolic Geometry. Euclidean Geometry began around 300 BC in Euclid's book *Elements*. Euclidean Geometry was based on five axioms, which were:

1. For every point *P* and for every point *Q* not equal to *P* there exists a unique line *l* that passes through *P* and *Q*.
2. For every segment AB and for every segment CD there exists a unique point *E* such that *B* is between *A* and *E* and segment CD is congruent to segment BE.
3. For every point *O* and every point *A* not equal to *O*, there exists a circle with center *O* and radius OA.

4. All right angles are congruent to each other.

5. For every line *l* and for every point *P* that does not lie on *l* there exists a unique line *m* through *P* that is parallel to *l* (Greenberg, 1993).

Throughout history, mathematicians have proposed different axiom systems to make Euclid's proofs more rigorous. "Quite a few of Euclid's proofs are based on reasoning from diagrams" (Greenberg, 1993, p. 70) David Hilbert worked with Euclid's axioms and added the betweenness axiom and the congruence axiom, which now brings the number of axioms to seven (Greenberg, 1993). The fifth axiom, or later called the parallel postulate has met with much controversy (Greenberg, 1993).

Mathematicians throughout history have tried to prove the fifth axiom using the first four axioms. Girolamo Saccheri and Johann Lambert both tried to prove the fifth axiom by assuming it was false and looking for a contradiction (Katz, 1993, p. 696). Lambert worked to further Saccheri's work by looking at a quadrilateral with three right angles (See Figure 1). Lambert

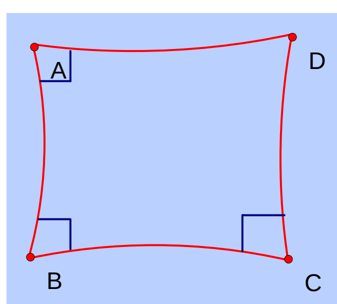


Figure 1

looked at three possibilities for the fourth angle: right, obtuse or acute. He figured that if he could disprove that it is an obtuse or acute angle then he would prove the parallel axiom.

However, Lambert was unable to disprove that the angle was acute (Katz, 1993,). Carl Gauss, Nikolai Lobachevsky and Janos Bolyai, whose work focused on a negation of the parallel postulate, worked at approximately the same time on what is now called Hyperbolic Geometry, where the fourth angle is acute (Katz, 1993, p. 692).

Returning to the definition of Hyperbolic Geometry, two parts were emphasized. First, the axioms of neutral geometry, which consist of the theorems that can be proven without the use of the parallel axiom, are included in Hyperbolic Geometry. Second, Hyperbolic Geometry includes a negation of the parallel axiom, the hyperbolic axiom. The hyperbolic axiom states that "in hyperbolic geometry there exist a line *l* and a point *P* not on *l* such that at least two distinct lines parallel to *l* pass through *P*"(Greenberg, 1993, p. 187). Please refer to figure 2. Negating the parallel postulate has led to another

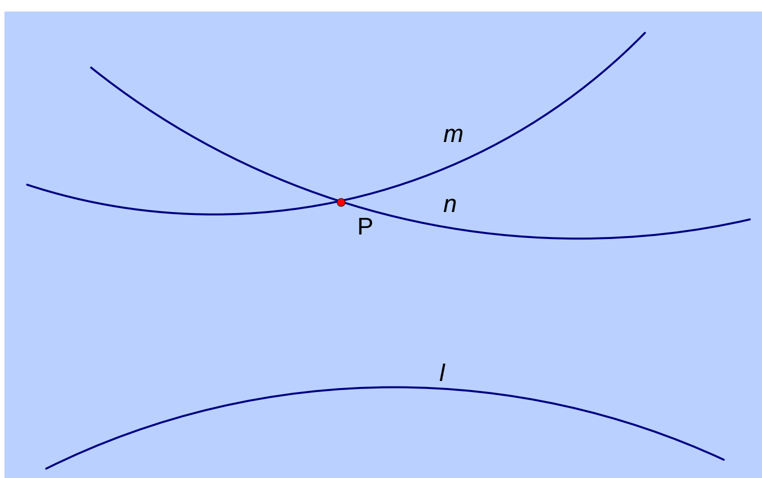


Figure 2

non-Euclidean geometry, Spherical Geometry.

Spherical Geometry is based on the following negation of the parallel postulate: given a line *l* and a point *P* not on the line there are no lines through point *P* parallel to line *l*. Spherical Geometry is useful because the surface for this geometry can be a sphere. "The typical study of geometry in modern classrooms works from the assumption that 'the land' to be measured is flat. . . . . We know, however, that our land is on the earth, which is, basically a sphere" (Heimer and Sharp, 2002). The lines in Spherical Geometry are great circles or circumferences on the sphere. The use of the globe and lines of longitude as a model of Spherical Geometry helps students to visualize Spherical Geometry (Heimer and Sharp, 2002). Spherical Geometry has many known applications such as flight paths for airplanes and meteorology, which give connections for the students as they are learning Spherical Geometry. However, the negation of the parallel postulate in Spherical Geometry causes numerous changes in the resulting geometry from Euclidean Geometry such as the definition of a line (Heimer and Sharp, 2002). On the other hand, Hyperbolic Geometry is more similar to Euclidean Geometry than Spherical Geometry because it only changes one axiom (Dwyer and Pfiefer, 1999).

There are some models to help visualize Hyperbolic Geometry. One such model of Hyperbolic Geometry is the disk model by Henri Poincare. One way of describing the Poincare disk model is

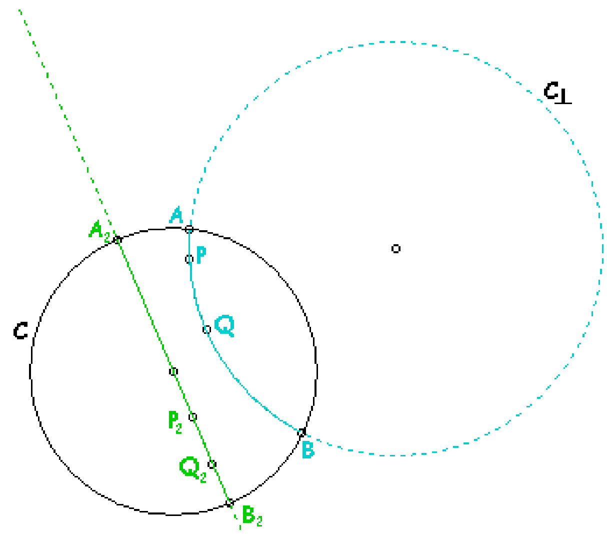


Figure 3 (Castellanos, 2002)

By taking an open hemisphere, that is, a hemisphere without its boundary. The points of the open hemisphere will be the points in our model. The straight lines will be the open hemicircles which are perpendicular to the boundary, similar to the slices of a halved onion when sliced perpendicularly on a breadboard (Lenart, 2004). In Figure 3, P1Q1 is a straight line since it is perpendicular to the boundary. Lines P1Q1 and P2Q2 are parallel lines since these lines do not intersect. This model represents a way of visualizing by "finding Euclidean objects that represent hyperbolic objects" (Greenberg, 1993,p. 226).

Models help students with their visualization while they are learning new mathematical concepts. Heimer and Sharp (2002) used spheres as models when teaching their students Spherical Geometry. According to Heimer and Sharp, "We knew that the students would have to be able to visualize lines, segments, angles, and polygons on a plane and easily identify properties and definitions of these concepts before moving their thinking onto the sphere” (p. 183). One of Heimer and Sharp’s students commented also on how “it was easier to learn about spheres’ on a ball than on paper” (p. 183). Alejandro Solano and Norma Presmeg (1995) did a study on how participants used images and their relationships while learning new geometrical ideas. Solano and Presmeg's conclusion from the study, ”suggests that imagery and visualization . . . . play a vital role in solving mathematical problems” .

Technology has been used in the creation of models to help students visualize Hyperbolic Space. The research on the use of technology in the classroom seems to show a small but positive impact on student achievement, both in general use and more specifically in visualizing Hyperbolic Geometry. The North Central Regional Educational Laboratory did a study on the effect on the use of educational software (McCabe and Skinner, 2003). The study by NCREL “determined a mean ‘effect size’ of 0.30 for a combined sample of 4,314 students, which suggested that teaching and learning with technology had a small but positive effect on student outcomes when compared with traditional instruction” (McCabe and Skinner, 2003). SRI International Center for Technology in Learning found a similar result after a meta-analysis of 31 studies that focused on student improvement in mathematics in connection with the use of educational software (McCabe and Skinner, 2003).

According to NCTM’s Principles and Standards, “The graphic power of technological tools affords access to visual models that are powerful but that many students are unable or unwilling to generate independently"(2000, p. 25). Geometer's Sketchpad can help with the visual models to which NCTM refers. Quinn and Weaver (1999) discuss specifically what role Geometer's Sketchpad can play in exploring a geometry problem:

> When students are given a geometric problem to solve the first thing they should do is represent the problem with a geometric model to help them visualize what the problem is asking. Then they need to form an idea about what they think the result will be. Geometer's Sketchpad is a great tool to allow the students to check their conjectures. By allowing the students to discover these geometric relationships, the students are learning important geometric thinking skills. As they discover these relationships they are able to develop many geometric properties on their own (p. 85).

Dynamic geometry such as Geometer's Sketchpad allows the teacher to turn the classroom into a science lab (Olive, 2000). According to Olive,

> Mathematics becomes an investigation of interesting phenomena, and the role of the mathematics student becomes that of the scientist: observing, recording, manipulating, predicting, conjecturing and testing, and developing theory as explanations for the phenomena.

Technology and models are tools that have opened the world of Hyperbolic Geometry. However, should high school geometry students study Hyperbolic Geometry when they have been studying and continue to study Euclidean Geometry? Euclidean Geometry explains nicely all the aspects of the plane and provides a good example of axiomatic mathematics. So what purpose would studying Hyperbolic Geometry fulfill for high school students? According to NCTM's Principles and Standards, "In grades 9-12 all students should establish the validity of geometric conjectures using deduction, prove theorems, and critique arguments made by others"(2000, p. 308). Studying the fifth postulate of Euclid would provide an excellent way to extend the validity of geometric conjectures. This was the main goal of the study that will be discussed in Chapter 3. "Hyperbolic geometry is usually introduced to students as a first example of a non-Euclidean geometry because it is the 'closest' one to Euclidean geometry, in that it involves changing only one axiom (Dwyer and Pfiefer, 1999). Teachers need to teach more than one system of mathematics. According to Lenart (2004), "modern mathematics, including geometry, knows very well that there are many systems possible. So the teacher should offer, not a fixed system, but a menu of systems" (p. 22). It has been thought that Euclidean Geometry represents the space we live in but Hyperbolic Geometry has brought to question whether the space we live in is curved or not. If the space were curved then Hyperbolic Geometry would explain the surface due to the curvature of Hyperbolic space. F. W. Bessel completed an experiment to find the sum of the angles in a triangle using the angle of parallax of a distant star (Greenberg, 1993). Due to experimental error, Bessel's experiment was inconclusive. "Because of experimental error, a physical experiment can never prove conclusively that space is

Euclidean – it can prove only that space is non-Euclidean"(Greenberg, 1993, p. 291). "Hyperbolic Geometry is a 'curved' space, and plays an important role in Einstein's General theory of Relativity" (Castellanos, 2002).

In summary, Hyperbolic Geometry includes a neutral geometry and a negation of the parallel postulate: There exists a line *l* and a point *P* not on the line such that there are at least two lines through point *P* that do not intersect the given line *l*. The use of models and technology help students in visualizing new mathematical concepts. Hyperbolic Geometry helps to explain curved space. In the next chapter, I will discuss the similarities and differences between Euclidean Geometry and Hyperbolic Geometry.

Chapter 2 – Hyperbolic Geometry

In this chapter, the similarities and differences between Euclidean Geometry and Hyperbolic Geometry will be discussed. First, an examination will be made of the axioms that form the basis for Hyperbolic Geometry. Next, the neutral theorems will be discussed. Finally, the theorems that might be affected by the negation of the parallel postulate will be studied.

Before examining the axioms of Hyperbolic Geometry, there are some concepts that need to be defined. One such concept is a metric space which is an ordered pair (M, d) where M is a non-empty set and d is a distance metric on M. Another concept is opposite rays, which are two distinct rays that emanate from the same point and are part of the same line (Greenberg, 1993, p. 16). Supplementary angles are two angles that share a common side, the other two sides are opposite rays, and therefore the angles are called supplements of each other (Greenberg, 1993, p. 17). The last concept is the definition of congruence. Two segments are congruent if the two segments have the same length. Two angles are congruent if the two angles have the same measure. Two triangles are congruent if the two triangles have all corresponding sides congruent and all corresponding angles congruent. The above definitions can be used to investigate and apply the following axioms of Hyperbolic Geometry.

According to Ramsey and Richtmyer (1995) there are seven axioms in Hyperbolic Geometry:

Axiom 1: If *A* and *B* are distinct points then there is only one line that passes through both of them.

Axiom 2: Let M, a metric space, be a non-empty set. Then a distance metric *d* is defined on M for all pairs of points *P* and *Q* so that the following hold:

a. *d*(P, Q)•0 for all P, Q ↲ M

b. *d*(P, Q)=0 if and only if P = Q

c. *d*(P, Q)=*d*(Q, P) for all P, Q ↲ M

d. *d*(P, R)□*d*(P, Q) + *d*(Q, R) for all P, Q, R ↲ M

Axiom 3: For each line *l*, there is a 1-1 mapping, *x*, from the set *l* to set of real numbers, such that if *A* and *B* are any points on *l*, then *d* (A, B) = $|x\ (A) - x\ (B)|$.

Axiom 4: If *l* is any line, there are three corresponding subsets HP_1 and HP_2, called half-planes, and the line *l* such that the sets *l*, HP_1 and HP_2 are disjoint and their union is all of the plane *P*, such that

a) if *P* and *Q* are in the same half-plane then segment *PQ* contains no point of *l*

b) if $\boldsymbol{P \in HP_1}$ and $\boldsymbol{Q \in HP_2}$ then segment *PQ* contains a point of *l*

Axiom 5: For each $\angle\vec{h},\vec{k}$ there is a number $\angle\vec{h},\vec{k}^{rad}$ in the interval [0, Π] called the radian measure of the angle, such that[1]

a) if $\vec{h}$ and $\vec{k}$ are the same ray then $\angle\vec{h},\vec{k}^{rad}=0$; if $\vec{h}$ and $\vec{k}$ are opposite rays then $\angle\vec{h},\vec{k}^{rad}=\div$

b) sum of the measure of the angle and its supplement equal Π

c) if $\vec{j}$ is in the interior of an $\angle\vec{h},\vec{k}$ then $\angle\vec{h},\vec{j}^{rad}+\angle\vec{j},\vec{k}^{rad}=\angle\vec{h},\vec{k}^{rad}$

d) if a ray $\vec{k}$ from a point *Z* lies in a line *l*, then in each half-plane bounded by *l*, the set of rays $\vec{j}$ from *Z* is in a 1-1 correspondence with the set of real numbers α in (0, Π) in such a way that α is equal to the $\angle\vec{j},\vec{k}^{rad}$ of $\angle\vec{j},\vec{k}$

e) if the ray $\vec{j}$ is determined as $\overrightarrow{ZP}$, where *P* is a point on $\vec{j}$ then angle α depends continuously on *P*

Axiom 6: If two sides and the included angle of a first triangle are congruent respectively to two sides and the included angle of a second triangle, then the triangles are congruent.

[1] The symbol $\angle\vec{h},\vec{k}$ represents the angle made of ray *h* and ray *k*. The symbol $\angle\vec{h},\vec{k}^{rad}$ represents the measure of the angle $\angle\vec{h},\vec{k}$.

> Axiom 7: There exists a line and a point not on the line such that there are at least two lines through that point that do not intersect the given line.

The first six axioms are the same in both Euclidean and Hyperbolic Geometry. The seventh one, more commonly known as the parallel postulate, is the only axiom which differs from that of Euclidean Geometry.

The negation of the seventh axiom leads to various consequences and thus differences between Euclidean Geometry and Hyperbolic Geometry. In Euclidean Geometry, there is only one line through a point not on a line that is parallel to the given line. Hyperbolic Geometry is based on one of the negations of that postulate. In Hyperbolic Geometry there are at least two lines that are parallel to the given line. Another subtle difference is in the sum of the angles in a triangle. According to Ramsey and Richtmyer, "the sum of the three interior angles of a triangle is less than or equal to 180" (1995, p. 48). They later go on to explain that in Euclidean Geometry the sum of the measures of the angles of a triangle equals 180 degrees while in Hyperbolic Geometry the sum of the measures of the angles of a triangle is less than 180 degrees (1995).

The axioms that form the basis of Hyperbolic Geometry have been given. The neutral theorems will now be examined. The theorems are called neutral because the theorems are the same in both Euclidean and Hyperbolic geometries. These theorems can be proved without the parallel postulate (Ramsey and Richtmyer, 1995, p. 30).

Neutral Theorems

Theorem 1: All vertical angles are congruent.

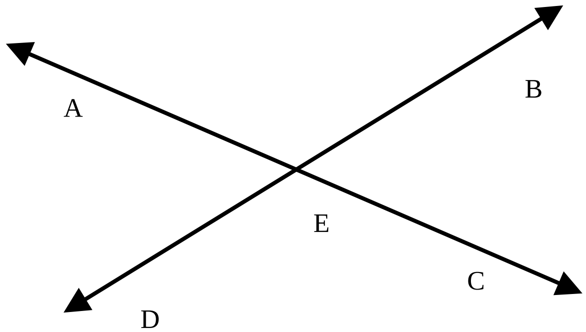


Figure 4

Let line AC and line BD intersect at point *E*. Therefore, ∠AEB is supplementary to ∠AED and ∠BEC. Supplement angles add to Π by Axiom 5. Since ∠AED and ∠BEC have the same supplement then they are congruent. Therefore vertical angles are congruent.

Theorem 2: An exterior angle of a triangle is greater than each of its nonadjacent interior angles.

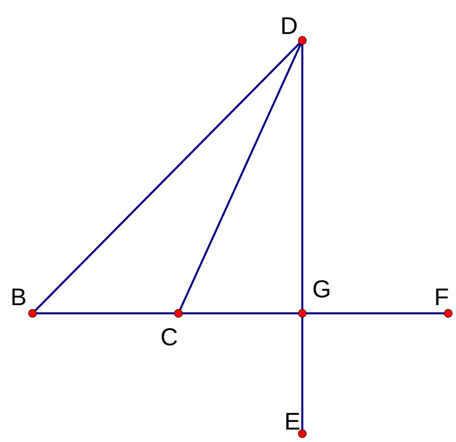


Figure 5

Referring to Figure 5, there are three cases for the relationship between ∨NDG and ∨DGF: m∨NDG ⁄ m∨DGF, m∨NDG < m∨DGF or m∨NDG > m∨DGF.

Case 1: m∨NDG ⁄ m∨DGF

Assume that m∨NDG ⁄ m∨DGF. So line MK is parallel to line NL because if alternate interior angles are congruent then parallel lines. However this contradicts the hypothesis that Figure 5 is a triangle.

Case 2: m∨NDG < m∨DGF

Assume that m∨NDG < m∨DGF. Draw ray DC so that m∨DGF ⁄ m∨GDC and crosses line NL. So, ray DC is parallel to line NL because if alternate interior angles are congruent then parallel lines. Line DC crosses line NL which contradicts the definition of parallel lines.

Since m∨NDG is not congruent to m∨DGF nor is m∨NDG greater than m∨DGF, then m∨NDG must be less than m∨DGF (Greenberg, 1993, p. 119).

Theorem 3: If two angles and the included side of a triangle are congruent to two angles and the included side of a second triangle then the triangles are congruent.

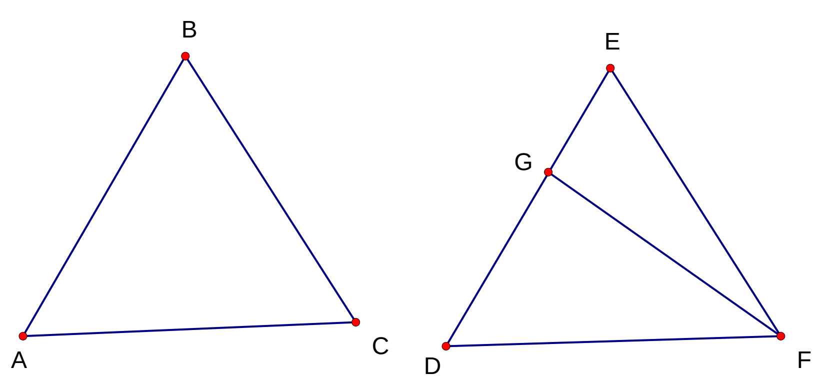


Figure 6

Referring to Figure 6, ∅ABC and ∅DEF are triangles with AC≅DF,∠BCA≅∠EFD and ∠BAC≅∠EDF. Plot G on ED so that AB ≅ GD, then ∅ABC ≅ ∅DGF by Axiom 6, Side-Angle-Side. Therefore, ∠BCA ≅ ∠GFD because corresponding parts of congruent triangles are congruent. So ∠GFD ≅ ∠DFE by the transitive property of equality. Since those angles are congruent then GF coincides with EF by axiom 5. So, ∅ABC ≅∅DEF. Therefore it can be proven that triangles are congruent by Angle-Side-Angle (Greenberg, 1993).

Theorem 4: If two angles and a non-included side of one triangle are congruent to two angles and a non-included side of another triangle then the two triangles are congruent.

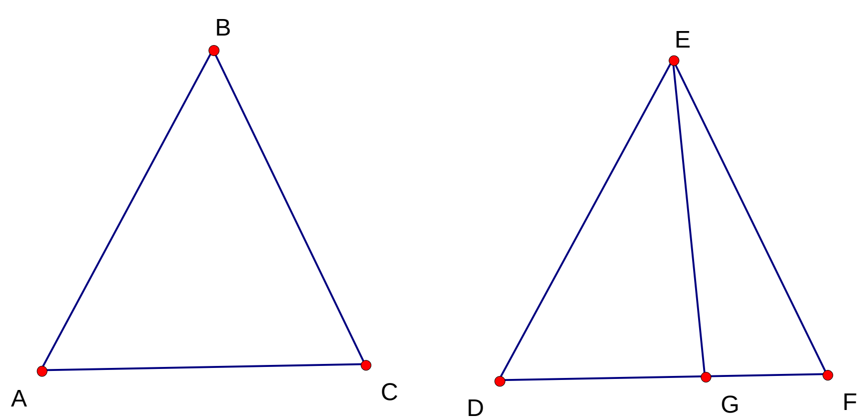


Figure 7

Referring to Figure 7, ∅ABC and ∅DEF are triangles with AB ≅ DE, ∠BAC ≅ ∠EDF and ∠ACB ≅ ∠DFE. Plot G on DF so that AC ≅ DG. Then ∅ABC ≅ ∅DEG by Side-Angle-Side. Therefore, ∠BCA ≅ ∠EGD because corresponding parts of congruent triangles are congruent. So, ∠EGD ≅ ∠EFG by the transitive property of equality. This means that GE and EF would not meet because if the corresponding angles of two lines are congruent then the lines are parallel lines, which are lines that do not intersect. This is a contradiction to the assumption that they do meet. So, G and F must coincide. Therefore, triangles can be proven congruent by Angle-Angle-Side (Ramsey and Richtmyer, 1995).

Theorem 5: If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse leg of another right triangle then the triangles are congruent.

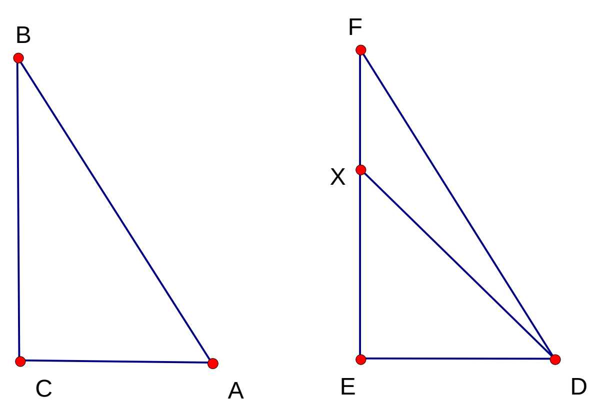


Figure 8

Referring to Figure 8, ∅ABC and ∅DEF are right triangles with AB ≅ DF, AC≅DE and m∨C=m∨E=90 degrees. Suppose that BC < FE. Find X on FE so that BC ≅ XE. Now ∅ABC≅∅DFX by Side-Angle-Side. Therefore, DX ≅ AB since corresponding parts of congruent triangles are congruent. So, DX ≅ DF by the transitive property of equality. All the points that are the distance AB away from D is a circle. FE will only intersect the circle twice, once on each side of DE. So, X and F must coincide. Therefore, triangles can be proven congruent through Hypotenuse-Leg (Ramsey and Richtmyer, 1995).

Theorem 6: The base angles of an isosceles triangle are congruent.

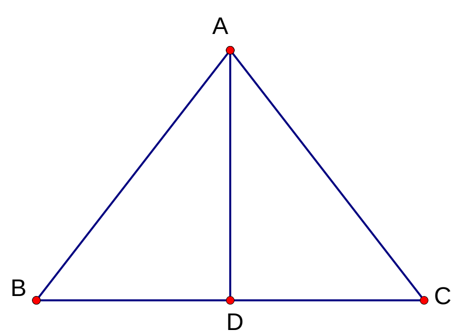


Figure 9

Let ∅**BAC** be an isosceles triangle with BA ≅ CA. Draw the altitude AD so that it meets side BC at a right angle. Then ∅BAD ≅ ∅CAD by Theorem 6. So, ∨ABD ≅ ∨ACD since all corresponding parts of congruent triangles are congruent. Therefore the base angles of an isosceles triangle are congruent.

Theorem 7: The altitude of an isosceles triangle bisects the vertex angle and the base.

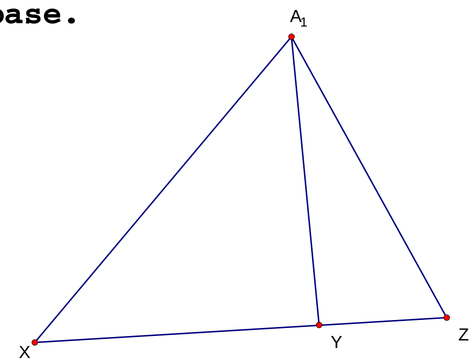


Figure 10

Let $\emptyset A_1XZ$ be an isosceles triangle with $XA_1 \cong A_1Z$. Draw altitude A_1Y from the vertex A_1 to the side XZ. Then $\angle A_1YX$ and $\angle A_1YZ$ are both right angles by the definition of an altitude of a triangle. Since base angles of an isosceles triangle are congruent, $\angle A_1XY \cong \angle A_1ZY$. Therefore, $\emptyset A_1XY \cong \emptyset A_1ZY$ because of Angle-Angle-Side or Hypotenuse-Leg. So, $\angle XA_1Y \cong \angle ZA_1Y$ and $XY \cong YZ$ because corresponding parts of congruent triangles are congruent. Therefore the altitude of an isosceles triangle bisects the vertex angle and the base.

Theorem 8: If a triangle is equilateral then it is equiangular.

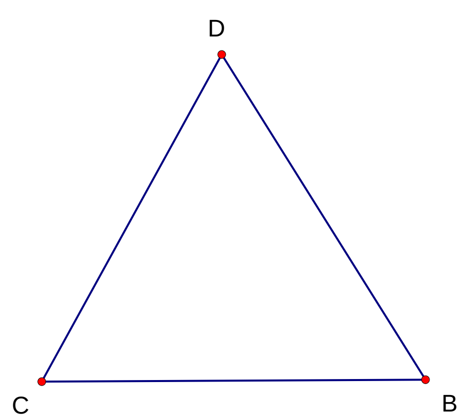


Figure 11

Let ∅CDB be an equilateral triangle. Then ∅CDB is an isosceles triangle with DC⁄DB. Therefore the base angles are congruent. So ∠DCB ≅ ∠DBC. Also, ∅BDC is also an isosceles triangle with BD⁄BC. Since the base angles are congruent, then ∠BDC ≅ ∠BCD. Therefore ∠DBC ≅ ∠DCB ≅ ∠BDC. This proves that if you have an equilateral triangle then you have an equiangular triangle.

Theorem 9: The longest side of a triangle is across from the greatest angle.

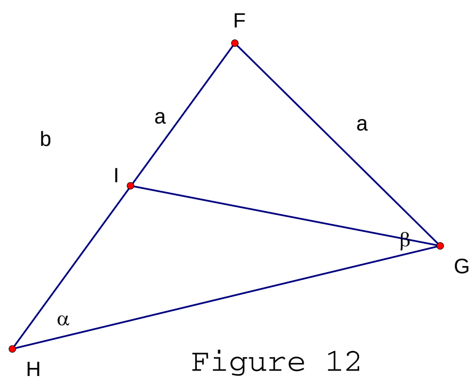


Figure 12

Referring to Figure 12, ∅FGH is a triangle with b = FH and a = FG. Let α =∠FHG and β = ∠FGH and assume a < b. Find I on HF so that IF = FG. So, ∅FIG is an isosceles triangle, which means that ∠FIG and ∠FGI are congruent. Therefore ∠FGH is greater than ∠FGI by axiom 5. Then ∠FIG is greater than ∠FHG by Theorem 2. Therefore, ∠FGH is greater than ∠FHG that is $\beta > \alpha$. Based on the assumption that a < b, the proven result that $\beta > \alpha$ and that β is opposite b and α is opposite a, the longest side of a triangle is opposite the greatest angle (Ramsey and Richtmyer, 1995).

Theorem 10: If three sides of one triangle are congruent to the corresponding three sides of another triangle then the triangles are congruent.

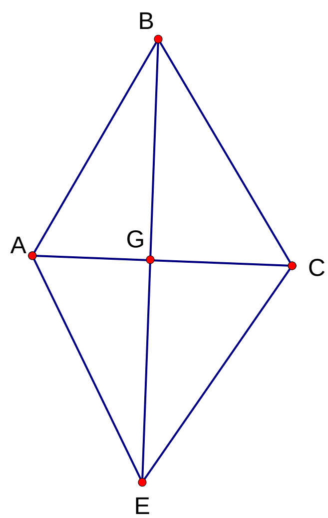


Figure 13

Referring to Figure 13, ∅ABC and ∅AEC are triangles with AB⁄AE, BC⁄EC and AC⁄AC. Then ∅BCE and ∅BAE are isosceles triangles. Therefore, ∨CBG⁄∨CEG and ∨ABG⁄∨AEG because of Theorem 6. So, ∨AEC⁄∨ABC by axiom 5. Therefore, ∅ABC⁄∅AEC by SAS.

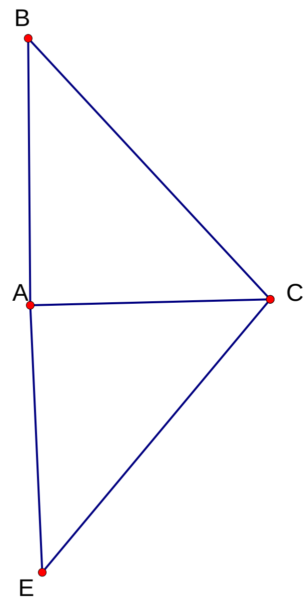


Figure 14

Referring to Figure 14, ∅ABC and ∅AEC are triangles with AB⁄AE, BC⁄EC and AC⁄AC. Then ∅BCE is an isosceles triangle. So, ∨ABC⁄∨AEC by Theorem 6. Therefore, ∅ABC⁄∅AEC by SAS.

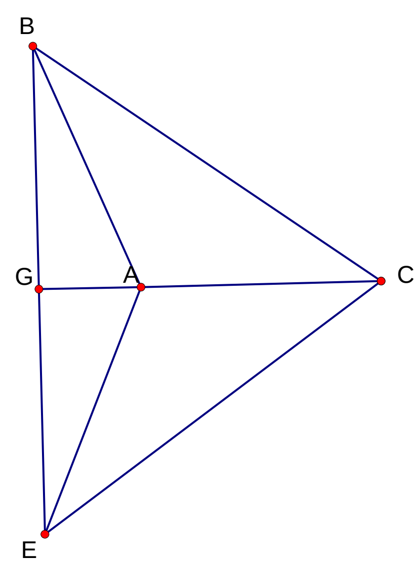


Figure 15

Referring to Figure 15, ∅ABC and ∅AEC are triangles with AB⁄AE, BC⁄EC and AC⁄AC. Then ∅BCE and ∅BAE are isosceles triangles. Therefore, ∨CBE⁄∨CEB and ∨ABE⁄∨AEB by Theorem 6. So, ∨CBA⁄∨CEA by axiom 5. Therefore, ∅ABC⁄∅AEC by SAS. Therefore, triangles with corresponding congruent sides are themselves congruent.

The proofs of the neutral theorems are identical to proofs of the analogous results in Euclidean Geometry. The proofs are identical because the proofs do not use the parallel postulate. The next theorems and their proofs may or may not be affected by the Hyperbolic parallel postulate. Therefore, the proofs may differ from those in Euclidean Geometry.

Theorems That Might Be Affected By the Hyperbolic Parallel Postulate

The study of quadrilaterals in Hyperbolic Geometry is one instance in which Hyperbolic Geometry differs from Euclidean Geometry. Rectangles, quadrilaterals having four right angles, do not exist in Hyperbolic Space. Quadrilaterals can be divided into two triangles, ⎝ABD and ⎝BCD by drawing one diagonal.

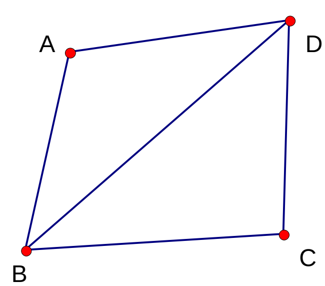


Figure 16

m∨ABD + m∨BDA + m∨DAB < 180ϒ

m∨DBC + m∨BCD + m∨CDB < 180ϒ

By angle addition:

m∨ABD + m∨DBC = m∨ABC

m∨BDA + m∨CDB = m∨CDA

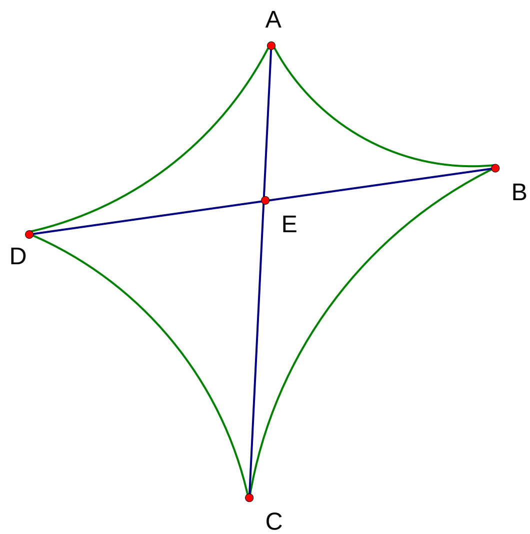
By adding the two inequalities:

m∨ABC + m∨BCD + m∨CDA + m∨DAB < 360ϒ

If three of the angles were 90 degrees then by the argument above, the last angle would be less than 90 degrees. Euclidean rectangles are quadrilaterals with four right angles. The sum of the angles of a rectangle equals 360 degrees. Therefore, rectangles do not exist in Hyperbolic Geometry. Therefore, squares, quadrilaterals having four right angles and four congruent sides, do not exist in Hyperbolic Geometry (Ramsey and Richtmyer, 1995). This, however, does not mean that there are no regular quadrilaterals in Hyperbolic Geometry. A regular quadrilateral is one for which all sides are congruent and that all angles are congruent. Regular quadrilaterals and rhombi exist in Hyperbolic Geometry because there is a model in which such figures exist. “The main property of any model of an axiom system is that all theorems of the system are correct in the model. This is because logical consequences of correct statements are themselves correct” (Greenberg, 1993, p. 52). These next proofs focus on the characteristics of rhombi and regular quadrilaterals in Hyperbolic Geometry.

Theorem 11: The diagonals of a rhombus bisect each other, are perpendicular, and bisect the angles of the rhombus.

Figure 17



Quadrilateral ABCD is a rhombus with all sides congruent and diagonals AC and DB. So, ∅ABD≅∅CBD and ∅ADC≅∅ABC by Side-Side-Side. Therefore, ∠DAB≅∠DCB and ∠ADC≅∠ABC. That is, the opposite angles of a rhombus are congruent. Continuing with the proof, ∠DAE≅∠BAE and∠ADE≅∠CDE since corresponding parts of congruent triangles are congruent. So, ∅DAE≅∅BAE and ∅ADE≅∅CDE by Side-Angle- Side. So, $\overline{AE} \cong \overline{EC}$ and $\overline{DE} \cong \overline{EB}$. Therefore, the diagonals of a rhombus bisect each other. Continuing further, ∠AEB≅∠AED since all corresponding parts of congruent triangles are congruent. These angles are also supplements since the angles share a side and the other two sides are opposite rays and therefore the measures sum to □. Therefore, the diagonals of a rhombus are perpendicular. Continuing, ∅AEB≅∅CEB and ∅DEC≅∅BEC by Side-Side-Side. So, ∠ABE≅∠CBE and ∠DCE≅∠BCE. Therefore, the diagonals of a rhombus bisect the angles of the rhombus.

Theorem 12: The diagonals of a regular quadrilateral bisect each other and they are perpendicular to each other.

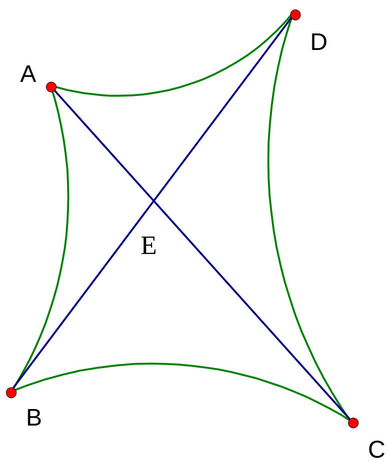


Figure 18

Regular quadrilaterals are a subset of rhombi. Therefore Theorem 12 has already been proven through Theorem 11. The following proof is specific to regular quadrilaterals. Quadrilateral ABCD is a regular quadrilateral with all sides congruent and all angles congruent with the diagonals AC and BD. By Side-Angle-Side, ∅BAD≅∅BCD and ∅ADC≅∅ABC. Therefore, ∠ABE≅∠CBE, ∠ADB≅∠CDB, ∠DAE≅∠BAE, and ∠DCE≅∠BCE. So, ∅AEB≅∅CEB and ∅DCE≅∅BCE by Side-Angle-Side. Therefore, $\overline{AE} \cong \overline{EC}$ and $\overline{BE} \cong \overline{ED}$. Therefore, the diagonals of a regular quadrilateral bisect each other. Continuing with the proof, ∠DEC≅∠BEC since corresponding parts of congruent triangles are congruent. These two angles are also supplementary angles, so their measures sum to Π. Since the angles are supplementary and congruent then the angles each have measure, 90 degrees and therefore are right angles. Therefore, the diagonals of a regular quadrilateral are perpendicular.

The question remains, do parallelograms exist in Hyperbolic Geometry. If parallelograms exist, what are their characteristics? Parallelograms are quadrilaterals in which both pairs of opposite sides are parallel, sides of a quadrilateral that never intersect. Since parallel lines exist in Hyperbolic Geometry, parallelograms exist in Hyperbolic Geometry. However, the characteristics of parallelograms in Hyperbolic Geometry differ from the characteristics of parallelograms in Euclidean Geometry. Before discussing the parallelograms, we discuss the alternate interior angle theorem.

The alternate interior angle theorem states that if the alternate interior angles of a pair of lines are congruent then the lines are parallel. The converse is also true in Euclidean Geometry. The converse is not necessarily true in Hyperbolic Geometry. Refer to the counterexample in figure 18. The diagram shows two lines, line AB and line CD, that are parallel because they never intersect. Angles ∨FAC and ∨ECA are alternate interior angles. These angles are not congruent. Therefore, the converse to the alternate interior angles theorem is not true in Hyperbolic Geometry. These angles would be congruent in Euclidean Geometry according to the converse of the alternate interior angle theorem.

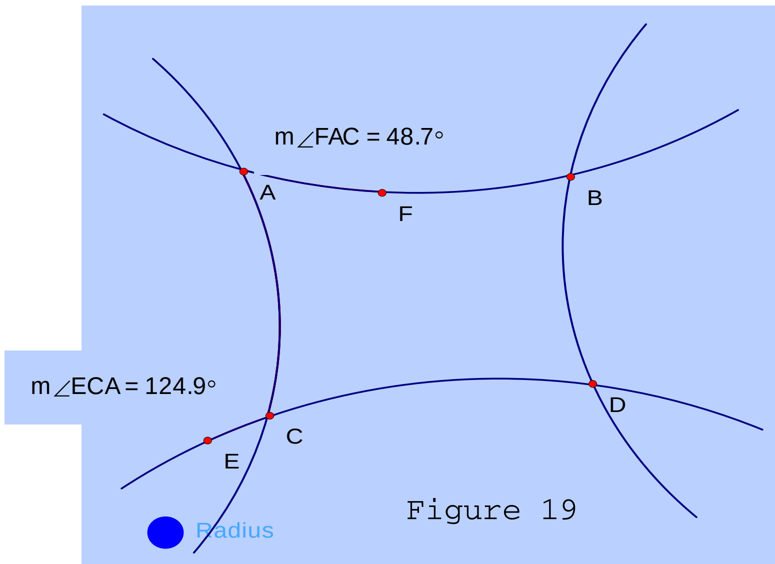


Figure 19

In Euclidean Geometry, the properties of parallelograms, such as the opposite sides are congruent or the opposite angles are congruent, are proved using the converse of the alternate interior angle theorem. Therefore, these results about parallelograms in Euclidean Geometry, cannot be proven in Hyperbolic Geometry.

The similarities and differences between Hyperbolic Geometry and Euclidean Geometry have been discussed. The similarities between Hyperbolic Geometry and Euclidean Geometry include the neutral theorems. The differences between Hyperbolic Geometry and Euclidean Geometry are the theorems that may or may not be affected by the Hyperbolic parallel postulate including theorems on regular quadrilaterals, rectangles, squares and rhombi. In the next chapter, the study of teaching Hyperbolic Geometry to high school students using technology will be discussed.

Chapter 3 – Hyperbolic Geometry in the High School Classroom

Usually students study only Euclidean Geometry in their high school geometry class. This paper will discuss what happened when high school students looked at non-Euclidean geometry and more specifically, Hyperbolic Geometry. The study was conducted with 51 ninth grade students who were enrolled in geometry. The students had been studying Euclidean Geometry for the first eight months of the school year. This study occurred for a week and a half in May, after the completion of the students' study of Euclidean Geometry.

The students began their study of non-Euclidean geometry with a two-day study of Spherical Geometry. The students completed an activity using tennis balls and rubberbands to model Spherical Geometry. The tennis ball represented the sphere and the rubberbands represented lines in Spherical Geometry, which

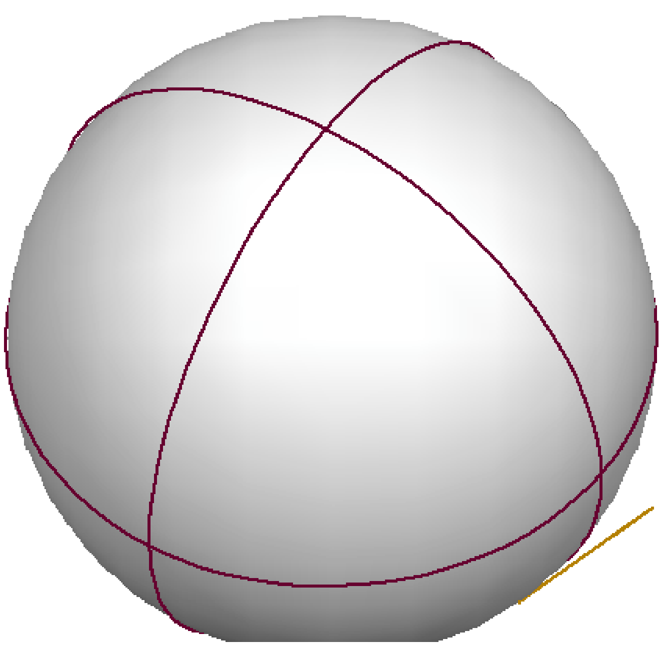


Figure 20

are great circles or circumferences on the sphere. Once the students accepted the new given definition of a line in this space, a circle whose center is the center of the sphere, the students investigated the existence of parallel lines, e.g., lines that never intersect (Figure 20). The students were able to see that parallel lines do not exist in Spherical Geometry. Next, the students investigated the sum of the angles of a triangle in Spherical Geometry. The students created triangles on the tennis ball using rubberbands, and then the students measured the angles by using protractors to approximate the angle measures. All the students were surprised to find that in the triangle the sum of the measures of the angles is greater than 180 degrees. Several students thought they had measured incorrectly because the angles did not add to 180 degrees exactly. The investigation examined triangles only and not quadrilaterals.

Following the investigations, there was a class discussion about the students' discoveries. The students spoke about the challenge in Spherical Geometry of understanding the definition of lines, the determination of the existence of parallel lines and realizing that the sum of the angles of triangles is not 180 degrees. This led to a brief conversation on the history of non-Euclidean Geometry and that there are other non-Euclidean Geometries. This conversation included a discussion of Euclid's five axioms. The discussion then led to the controversy of Euclid's fifth postulate. This opened the window for discussing the parallel postulate and the negations of the parallel postulate. The students then wanted to know if there were other negations of the parallel postulate and if Spherical Geometry was only one possible negation of the parallel postulate. The students' curiosity was now peaked about what the other non-Euclidean geometries might look like.

The students took a pre-test before they started their investigation of Hyperbolic Geometry. The purpose of the pre-test and post-test was to see if the lesson was effective in helping students to learn some differences between Euclidean Geometry and Hyperbolic Geometry. The test was a list of 34 theorems. The students had to label whether each theorem worked only in Euclidean Geometry, only in Hyperbolic Geometry, in both geometries, or in neither geometry. (A copy of the test is available in Appendix C.)

After administering the pre-test, the students spent two days working in groups of 3-4 in the computer lab completing activities on the website, NonEuclid: Interactive Constructions in Hyperbolic Geometry (Castellanos). This website contains a Java applet that has a Geometer's Sketchpad-like atmosphere but in hyperbolic space using the Poincare Model. Before this, students had had used Geometer's Sketchpad throughout the school year. Students had used Geometer's Sketchpad to investigate quadrilaterals in Euclidean Geometry. The students had also used Geometer's Sketchpad to investigate centers of triangles and reflections and rotations in Euclidean Geometry.

With the Hyperbolic Geometry applet, the students drew lines, line segments, rays, and circles. The applet allowed students to draw any polygon they wanted. Students were also able to measure segment lengths and angle measures. Students used the applet to test the Euclidean Geometry theorems listed in the activities. Appendix B contains a copy of the activities.

The students completed the activities while working together, questioning, discussing and explaining their thoughts to each other. Certain investigations challenged the students in their thinking. One example is that two triangles may not "look" congruent in Hyperbolic Geometry but could be proven to be congruent. Other issues arose while investigating parallelograms in Hyperbolic Geometry. The students could not seem to get past their picture of what a parallelogram looks like in Euclidean Geometry (See Figures 21 and 22). The students believed that the

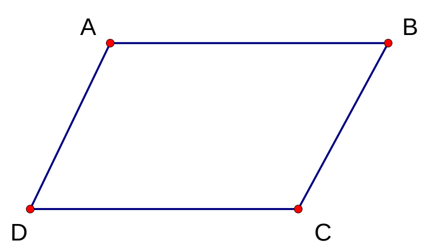


Figure 21

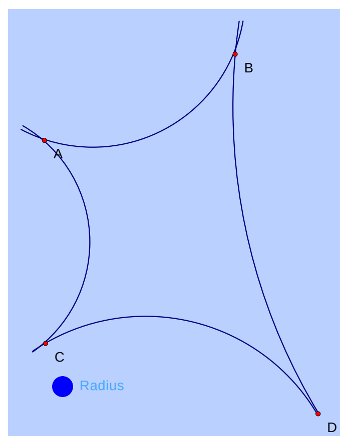


Figure 22

properties of parallelograms, such as opposite sides are congruent and opposite angles are congruent, would still work because they were thinking of the Euclidean picture of a parallelogram. The students were redirected to thinking about the definition of a parallelogram, a quadrilateral with both pairs of opposite sides parallel. Then the students were asked to think about the definition of parallel lines, two lines that never intersect. This opened the possibilities of parallelograms and the consequences of parallelograms in Hyperbolic Geometry. The students were able to accept that parallelograms exist in Hyperbolic Geometry but that the Euclidean properties do not hold in Hyperbolic Geometry.

The students were able to use their knowledge of results about quadrilaterals to help them as well. The students quickly realized that if rectangles do not exist in Hyperbolic Geometry then neither do squares. However, the students were confused by the aspect of a regular quadrilateral existing when a square does not. The students were again redirected to think about the definition of a regular quadrilateral, a quadrilateral with four congruent sides and four congruent angles. The redirection helped to rekindle their curiosity. The students did seem to resolve the issue of regular quadrilaterals, which will be, discussed later in the students' test results.

The questions and discussions that students had in their groups led to a whole class discussion the next day. The students brought up the issues that the individual groups had while they were working on the activities but then went further with the topics. The students started with the topic of the congruent triangles and why some congruent triangles did not "look" congruent. The students, after some discussion, agreed

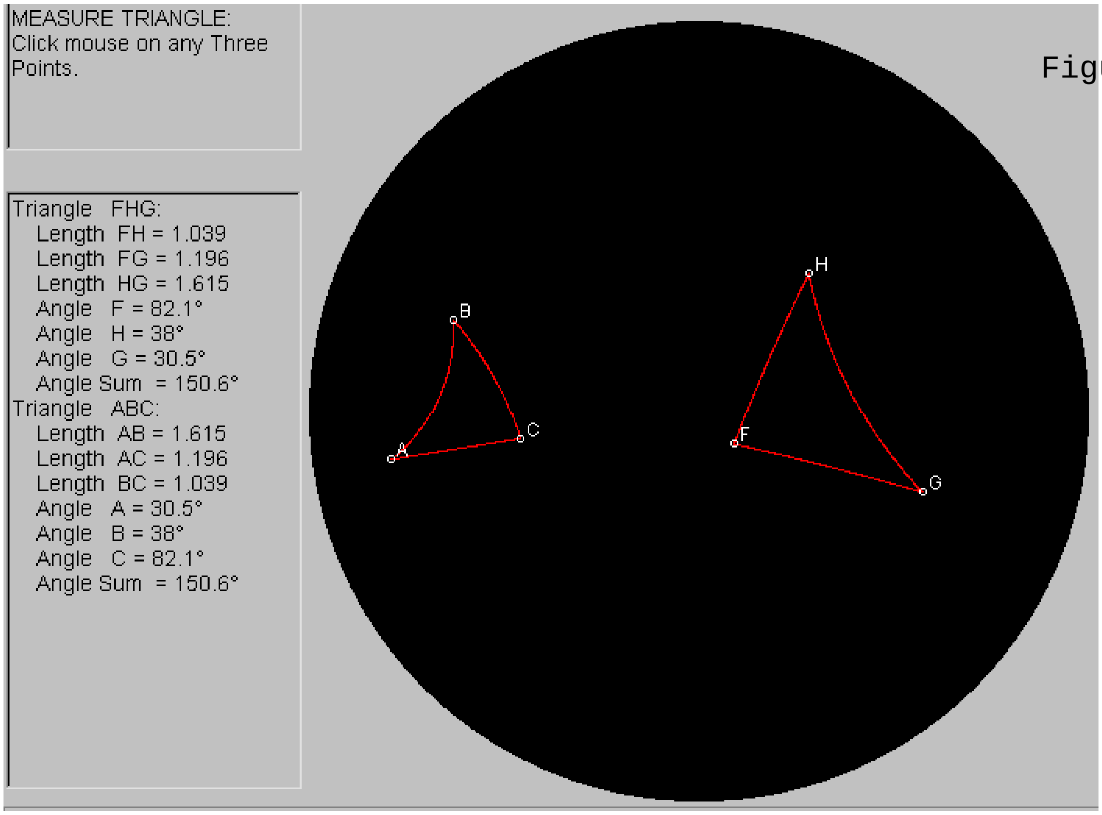


Figure 23

that since the space is curved the triangles did not "look" congruent (See Figure 23). The students discussed the difficulty of using the computer Poincare model because it looked flat when it was representing a curved space.

The students had a hard time overcoming their view of what parallel lines are. The students were focused on the picture of two lines that look like railroad tracks versus the definition, which says that parallel lines are two lines that never intersect. Each time I asked students what parallel lines are, they would show draw two lines like railroad tracks on their paper. The students were not seeing how there could be parallel lines in Hyperbolic Geometry since the lines are curved. Once they remembered the definition, they more easily accepted parallel lines that did not "look" parallel.

The discussions during and after the activity indicated the students' difficulty in ignoring definitions and consequences of theorems that they had learned and trusted throughout the school year in Euclidean Geometry with the perspective of Hyperbolic Geometry. The students were asking "Why?" questions such as, "Why is the space curved?" "Why don't rectangles exist in Hyperbolic Geometry?" These questions provide anecdotal evidence that the goal for students to better question why something does or does not exist had been met. To determine the students' knowledge and ability to distinguish between Euclidean Geometry and Hyperbolic Geometry, students completed the post-test, which was the same as the pre-test (Appendix C).

The null hypothesis is that there is no difference between the mean of the pre-test and the mean of the post-test. The alternative hypothesis is that the mean of the post-test is greater than the mean of the pre-test. I predicted that the mean score of students' post-test would exceed the mean score of students' pre-test. The data and graphs of student performance are provided in Appendix D. The mean, median and mode all increased from the pre-test to the post-test (See Table 1).

	Pre-Test	Post-Test
Mean	15.0588	21.5490
Median	15	22
Mode	18	22

Table 1

Of 34 questions, the mean of students' post-test score was 21.5490, greater than the mean of students' pre-test score of

15.0588. The test statistic ($t= -9.260$, $n=51$) exceeds the critical value of –2.407. The 99% Confidence Interval of the difference of the means was (-8.36708, -4.61331). The confidence interval of the difference of means does not include 0. Therefore, I would reject the null hypothesis. The mean of the post-test is statistically significantly greater than the mean of the pre-test. The probability that the observed difference between the means ($\sum d= -6.49020$) would have occurred by chance if the null hypothesis were true is less than 0.01. Therefore, I have concluded that the lesson, with its use of technology, was effective in helping students learn about Euclidean Geometry versus Hyperbolic Geometry.

Though the lesson was effective, there are some changes that I would make the next time that I taught the lesson. The students seemed to understand the main difference between Hyperbolic Geometry and Euclidean Geometry. According to the post-test results (See Appendix D), most of the students answered the questions correctly about the parallel postulate and the negation of the parallel postulate. All the students were able to say that the sum of the measures of the angles in a triangle equals 180 degrees is true only in Euclidean Geometry. However, the students had difficulty with ideas that challenged their mental picture and focused on the use of the definition. One example is equilateral triangles are also equiangular. The students seemed to understand that the angles in an equilateral triangle are not each 60 degrees but could not see that the angles could still be congruent. The students seemed to have the same difficulty with regular quadrilaterals. Next time, I would spend more time talking about the Euclidean definitions of the objects that the students were going to investigate. Hopefully, this would allow the class to look at their own mental models and would alleviate some of the confusions that happened during the activity. This would also allow the students to concentrate on the properties of the geometric objects and the relationship of the objects.

The goal for this study was for students to analyze and evaluate versus remember and recall previously learned concepts. Hyperbolic Geometry was selected over other non-Euclidean geometries due to the number of similarities to Euclidean Geometry. In addition, the activity provided students with the opportunity to see that other geometries exist that are as valid as Euclid's geometry. Since the hyperbolic space is so abstract, the use of technology was implemented to help students visualize the space. The technology provided a model that students could manipulate and see what happens in hyperbolic space. Class discussion revealed students' observations in working with hyperbolic space and the difference between Hyperbolic Geometry and Euclidean Geometry. The students were examined before and after the lesson to determine the lesson's effectiveness. Based on the test results, the lesson was judged effective.

Hyperbolic Geometry used to be an abstract idea that was out of reach for high school geometry students. Now, models and technology help students to visualize Hyperbolic Geometry. The non-Euclidean geometries were discovered years after Euclid's geometry. By sparking an interest in high school students, there may be more discoveries yet in the field of geometry. I will continue to teach non-Euclidean geometries to show my students that there are multiple ways to look at mathematics.

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