Learning Objectives for Math 166

Chapter 6 — Applications of Definite Integrals

Section 6.1: Volumes Using Cross-Sections

- Draw and label both 2-dimensional perspectives and 3-dimensional sketches of the solid whose volume is being calculated. The 2-dimensional sketches should include the cross-sections. The 3-dimensional sketches should include a "slice" as well as the entire solid.
- Consider the axis of revolution and determine how to partition the interval (whether along x or the y axis.)
- Partition the solid and apply basic geometry to determine the volume of a "slice."
- Form a Riemann Sum and use the limit of the Riemann sum to set up the definite integral to calculate the volume of the solid.
- Evaluate the resulting definite integral.

Section 6.2: Volumes Using Cylindrical Shells

- Draw a 2-dimensional perspective of the region to be rotated about the axis of revolution and partition the region to be rotated based on its position in relation to the axis of revolution (whether along the x or the y.)
- Draw and clearly label 3-dimensional sketches of the solid whose volume is being calculated. The 3-dimensional sketches should include the entire solid and a "cylindrical shell" resulting from a rectangular piece of the partition rotated about the axis of revolution.
- Use basic geometry to approximate the volume of the cylindrical shell. Set up a Riemann Sum of the volumes of the cylindrical shells. Use the limit of the Riemann Sum to set up the definite integral to give the volume of the solid.
- Evaluate the resulting definite integral.

Section 6.3: Arc Length.

- State the definition for arc length and be able to explain how to rewrite the formula for arc length in terms of $\frac{dy}{dx}$ or $\frac{dx}{dy}$, or the arc length differential ds.
- Explain how the formula for arc length develops from the Pythagorean Theorem after a curve has been partitioned into small pieces and approximated using linear segments.
- Explain how the definite integral arises from partitioning the curve into small pieces, approximating the lengths of the small curved pieces using lengths of short linear segments, and then summing the lengths of the short linear segments to then develop the definite integral.

Section 6.4: Areas of Surfaces of Revolution

- Explain how the concept of partition and sum applies to the areas of surfaces of revolution.
- Visualize and sketch the surface generated by revolving a graph of a function about an axis.
- Explain how the arc length differential contributes to the formula for the area of a surface generated by revolving a graph of a function about an axis. Use the arc length differential in conjunction with the partitioned interval and the ?radii? of the resulting bands to set up the Riemann Sum and, ultimately, the definite integral.

Section 6.5: Work and Fluid Forces

- State the definition for Work done by a constant force: $W = Fd$. Be able to work with the appropriate units, both in the English system and the metric system.
- Understand and explain what is meant by a "variable force acting in a line." For such problems demonstrate cutting the line into small pieces, calculating (approximating) the work done over each small distance, then summing these "small works" to approximate the total work, and finally passing to a Riemann Integral to calculate the exact value of the work.
- State and explain, with a diagram, Hooke's Law. As an example of the previous bullet, calculate the work done in stretching or compressing a spring.
- Explain and work with a second approach to calculate the work done in moving a large mass by cutting the mass into small pieces, then calculating the work done in moving each small piece, summing the "small works" for these pieces, then passing to a Riemann Integral to calculate the exact value of the work. Examples of such problems involve pumping a fluid from one container to another.

Section 6.6: Moments and Centers of Mass

- Calculate the moment and center of mass of a system of point masses on a line.
- Calculate the moments about the x- and y- axes and the center of mass of a planar system of point masses.
- Given a collection of rectangles in the plane, calculate the center of mass of the collection of rectangles, given that you know the mass and center of mass of each rectangle.
- Given a region in the plane, say between two curves, demonstrate how to partition the region into rectangles, find the mass and center of mass of each rectangle, then calculate the center of mass of the system of rectangles. Finally, by letting the

norm of the partition approach 0, arrive at the integral expressions for the x- and y-coordinates of the center of mass.

Chapter 8 — Techniques of Integration

Section 8.1: Integration by Parts.

- Understand and explain the mechanics of integration by parts as seen in the two forms

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx \qquad \text{or} \qquad \int u\,dv = uv - \int v\,du.$$

- Given an integral, describe the features of the integral that suggest it might be done by applying integration by parts.
- Given an integral to be evaluated by integration by parts, explain strategies for selecting u and dv.
- Apply integration by parts two or more times, as needed, to evaluate an integral.
- Sometimes,when evaluating an integral I using integration by parts, the integral I appears again later in the work allowing you to solve for I. Be able to explain and demonstrate this process when appropriate, for example in evaluation $\int e^{ax}\cos(bx)\,dx$.

Section 8.2: Trigonometric Integrals.

- Know and work with the basic trigonometric identities needed in integration of trigonometric expressions:

$$1-\sin^2\theta = \cos^2\theta, \qquad 1+\tan^2\theta = \sec^2\theta, \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}, \qquad \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

- State and work with the derivatives and antiderivatives of the trigonometric functions $\sin(a\theta)$, $\cos(a\theta)$, $\tan(a\theta)$, $\cot(a\theta)$, $\sec(a\theta)$, $\csc(a\theta)$, $\sec^2(a\theta)$, $\csc^2(a\theta)$, $\sec(a\theta)\tan(a\theta)$, $\csc(a\theta)\cot(a\theta)$.
- Integrate expressions of the form $\int \sin^m(a\theta)\cos^n(a\theta)\,d\theta$ for all cases of nonnegative integers m and n.
- Integrate expressions of the form $\int \tan^m(a\theta)\sec^n(a\theta)\,d\theta$ for m an odd positive integer and/or n an even nonnegative integer

Section 8.3: Trigonometric Substitutions.

- Explain and use the relationship of some quadratic forms to trigonometric substitutions and identities:

$$\sqrt{a^2-x^2} \quad\longleftrightarrow\quad x = a\sin\theta \quad\longleftrightarrow\quad \sqrt{a^2-(a\sin\theta)^2} = |a\cos\theta|$$

$$\sqrt{a^2+x^2} \quad \longleftrightarrow \quad x = a\tan\theta \quad \longleftrightarrow \quad \sqrt{a^2+(a\tan\theta)^2} = |a\sec\theta|$$

$$\sqrt{x^2-a^2} \quad \longleftrightarrow \quad x = a\sec\theta \quad \longleftrightarrow \quad \sqrt{(a\sec\theta)^2-a^2} = |a\tan\theta|$$

- Demonstrate the ability to identify an appropriate trigonometric substitution, carry out the substitution, evaluate the resulting trigonometric integral, then reverse the substitution to express your answer in terms of the original variable. This can include some work with inverse trigonometric functions.

Section 8.4: Integration of Rational Functions by Partial Fractions.

- Given a rational function with a completely factored denominator, write down the form for the partial fractions expansion. (See the boxed table on Pages 472, 473.)
- Understand and carry out the algebra needed to find the values of the unknowns in a partial fractions expansion.
- Recognize the circumstances under which polynomial long division is needed, and carry out long division in such cases.
- Explain the process of integrating a rational function: do a partial fractions decomposition; integrate the individual fractions with linear denominators; use trigonometric substitution to change the fractions with quadratic denominators to trigonometric expressions; integrate the trigonometric expressions; reverse substitutions to produce an answer with the original variable.

Section 8.6: Numerical Integration.

- Explain what is numerical integration and describe circumstances in which you might need numerical integration.
- Draw a diagram to illustrate the trapezoid rule. Use the diagram to develop the trapezoid rule as seen on page 488.
- With the aid of your calculator, apply the trapezoid rule to approximate $\int_a^b f(x)\,dx$ for a simple function f.
- Explain and work with the error estimate for the trapezoid rule.

Section 8.7: Improper Integrals

- Determine if an integral is improper by identifying if one or both of the limits of integraion is infinite or if a vertical asymptote lies in the interval of integration.
- For an improper integral of either type, rewrite the improper integral as a limit.
- Given an improper integral, determine whether it converges or diverges by evaluation, the direct comparison test, or the limit comparison test.

Chapter 10 — Infinite Sequences and Series

Section 10.1: Sequences.

- Express a sequence as a list (with a generic term) and as a formula.
- Explain what it means for a sequence to converge and for a sequence to diverge.
- Calculate the limit of a sequence and apply theorems about limits, including L?H?opital?s Rule, to calculate more complicated limits.
- Determine whether or not a sequence is monotone.
- Explain the convergence/divergence options for a monotonic sequence and explain why such a sequence cannot diverge by oscillation.

Section 10.2: Infinite Series.

- Given an infinite series $\sum_{k=1}^{\infty} a_k$ explain the meaning of "terms of the series" and the ?sequence of partial sums? of the series.
- Given an infinite series $\sum_{k=1}^{\infty} a_k$, define what it means for the series to converge and what it means for the series to diverge.
- State the definition of geometric series. Be able to determine the n^{th} partial sum of any geometric series. Give conditions under which a geometric series converges and determine the sum of a convergent geometric series.
- Explain the meaning of "telescoping series" and give an example of a telescoping series. State conditions under which a telescoping series converges.
- Explain the relationship between the "n^{th} term divergence test" and Theorem 7 in the book: "If $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$."
- Explain why $a_k \to 0$is not enough reason to conclude that $\sum_{k=1}^{\infty} a_k$ converges.
- Explain why altering the first several terms of an infinite series does not affect the convergence or divergence of the infinite series.

Section 10.3: The Integral Test.

- Apply the Integral Test to determine if a series consisting of positive, decreasing terms converges or diverges.
- Given a series $\displaystyle\sum_{k=1}^{\infty} a_k$ for which $a_k = f(k)$ with f a positive, decreasing continuous function, draw the related graph to demonstrate the use of the integral test in which the terms a_k are represented as the areas of rectangles over or under the graph of the function f, depending on whether one is trying to prove divergence or convergence of the series.

- Given a series $S = \sum_{k=1}^{\infty} a_k$ and partial sum $s_n = \sum_{k=1}^{n} a_k$ for which $a_k = f(k)$ and f is positive, decreasing and continuous, use improper integrals to calculate upper and lower bounds for the remainder $R_n = S - s_n$.

Section 10.4: Comparison Tests.

- Know the conditions under which you may use the comparison test. Apply the comparison test to determine the convergence and divergence of infinite series.
- Know the conditions under which you may use the limit comparison test. Apply the limit comparison test to determine the convergence and divergence of infinite series.

Section 10.5: The Ratio and Root Tests.

- Explain the conditions under which the ratio test and/or the root test can be used to test for convergence of an infinite series.
- Describe the outcomes of the ratio and root test that allow you to conclude a series converges and the outcomes that allow you to conclude divergence. For what outcomes is the ratio test inconclusive? Give examples of series that demonstrate each conclusion.

Section 10.6: Alternating Series, Absolute and Conditional Convergence.

- State the three conditions for convergence of an alternating series by the Alternating Series Test.
- Explain what it means for an infinite series to converge absolutely.
- Explain what it means for an infinite series to converge conditionally (but *not* absolutely.)
- Given a series that converges by the Alternating Series Test, estimate the error when a partial sum of the series is used to estimate the sum of the series.

Section 10.7: Power Series.

- Explain what a power series is, including the meaning of the center of the power series and the coefficients of the power series.
- Explain clearly the meaning of radius of convergence for a power series and interval of convergence for a power series.
- Explain the role of the ratio test in finding the radius of convergence and the interval of convergence for a power series.
- Given a power series, apply the operations of algebra and calculus to the power series to obtain other power series.

Section 10.8: Taylor and Maclaurin Series..

- Given a function f that is infinitely differentiable at a point a, write down the general formula for the Taylor Series for f centered at $x = a$.
- Given a power series $\sum_{k=0}^{\infty} a_k(x-a)^k$ that converges to a function f on an interval $(a-r,\, a+r)$, be able to calculate any derivative $f^{(k)}(a)$.
- Given a function f calculate a given Taylor polynomial for f at a given point a.
- Derive the Maclaurin series for $\sin x$, $\cos x$, e^x, and $\frac{1}{1-x}$.

Section 10.9: Convergence of Taylor Series..

- Determine the n^{th} order Taylor polynomial, $P_n(x;a)$, about a specified center a, and calculate a corresponding error approximation, $R_n(x;a)$ on a given interval centered at a.
- Determine the power series and their intervals of convergence of new functions based on the known power series $\left(e^x, \sin(x), \cos(x), \text{and } \dfrac{1}{1-x}\right)$.

Section 10.10: The Binomial Series and Applications of Taylor Series.

- Develop the binomial series for expressions taking the form $(1+x^k)^m$.
- Use power (Taylor) series to approximate definite integrals of non-elementary functions and to evaluate limits taking indeterminate forms.
- Gain initial exposure to Euler's Identity.

Chapter 11— Parametric Equations and Polar Coordinates

Section 11.1: Parametrizations of Plane Curves

- Define the terms that arise in this Section: parametric curve, parametric representation, parameter, initial point, terminal point, parametrized, parametrization.
- Given a simple pair of parametric equations, plot several points for and graph the related curve.
- Given parametric equations for a simple curve (line, circle, parabola, hyperbola) perform the algebra to convert the representation to a Cartesian representation and classify (name) the curve.
- Given a description of a curve like a line, circle, ellipse, parabola, etc., write parametric equations for the curve.
- Describe the process through which a cycloid is generated. Derive parametric equations for the cycloid using as parameter the angle of rotation of the generating circle.

Section 11.2: Calculus with Parametric Curves

- Given a set of parametric equations for a curve and a point on the curve, find the slope of the line tangent to the curve at that point and the equation of the tangent line at that point.
- Given a set of parametric equations for a curve, find the area of a region bounded by the curve using the techniques of Chapter 5 and the substitution $x = x(t)$.
- Given a set of parametric equations for a curve, find the length of the curve.
- Relate the arc length differential as seen in Chapter 6 to the arc length differential as it arises in the context of parametric equations.

Section 11.3: Polar Coordinates

- Given polar coordinates (r, θ) for a point, graph the point in the plane. Explain and demonstrate how points are plotted for negative values of r.
- Given polar coordinates (r, θ) explain why $(-r, \theta + \pi)$ and $(r, \theta \pm 2\pi)$ describe the same point.
- Graph the fundamental equations $\theta = \alpha$ and $r = a$.
- Given an expression in polar coordinates, convert it to an equivalent expression in Cartesian coordinates.
- Given an expression in Cartesian coordinates, convert it to an equivalent expression in polar coordinates.

Section 11.4: Graphing in Polar Coordinates

- Given a formula $r = f(\theta)$ describing a curve C in polar coordinates, explain why the parametric equations

  $$x = x(\theta) = f(\theta)\cos\theta, \qquad y = y(\theta) = f(\theta)\sin\theta$$

  are parametric equations for C. Use this parametric representation to calculate slopes and arc lengths for parametric curves.
- Given that a point (r, θ) satisfies an equation $r = f(\theta)$, replace theta by $-\theta$, $\pi - \theta$, and $\pi + \theta$ and use the output to determine symmetries of the graph of $r = f(\theta)$.
- Given a polar equation $r = f(\theta)$, calculate r for several values of θ, plot the resulting points (r, θ) and use these to sketch a graph of the polar equation.
- Given a polar equation $r = f(\theta)$, plot the graph of $r = f(\theta)$ in a Cartesian system with horizontal axis θ and vertical axis r, then use this graph as a guide in graphing $r = f(\theta)$ in a polar system.
- Use your calculator to produce the graph of a given polar equation.

Section 11.5: Areas and Lengths in Polar Coordinates

- Derive and explain the formula for the area of a sector of angle α in a circle of radius a.
- Demonstrate how a region defined through polar equations is partitioned into sectors. Estimate the area of each piece using the "circular sector area formula" and use these areas to write a Riemann Sum to approximate the area of the region.
- Set up and evaluate integrals for areas bounded by curves described by polar equations.
- Working from the parametric equations

  $$x = x(\theta) = f(\theta)\cos\theta, \qquad y = y(\theta) = f(\theta)\sin\theta$$

  associated with a polar equation $r = f(\theta)$, demonstrate that the arc length differential

  $$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = \sqrt{(f(\theta))^2 + (f'(\theta))^2}\,d\theta = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

- Set up and evaluate integrals for lengths of curves described by polar equations.