Tangent planes and other miscellany

We have seen tangent planes done in two different ways. When we did differentiability for a function \( z = f(x, y) \) we said that a function locally looks like a plane along with some possible error, the plane we got was as follows:

\[
z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y-y_0).
\]

The second way we have seen these tangent planes is when dealing with \( F(x, y, z) = k \), which we can think of as a level surface. In this case for a point \( p = (x_0, y_0, z_0) \) we can use the properties of gradient to note that \( \nabla F(p) \) will be our normal vector so that our tangent plane is

\[
\nabla F(p) \cdot (x-x_0, y-y_0, z-z_0) = 0.
\]

or if we expand out the above we get the following:

\[
\frac{\partial F}{\partial x}(p)(x-x_0) + \frac{\partial F}{\partial y}(p)(y-y_0) + \frac{\partial F}{\partial z}(p)(z-z_0) = 0.
\]

It is important to note that these two definitions are compatible, i.e., if we wanted the tangent plane for \( z = f(x, y) \) we would get the same plane as if we worked with the function \( F(x, y, z) = f(x, y) - z = 0 \).

We can rearrange our terms above to get the following:

\[
\Delta z \approx f_x \Delta x + f_y \Delta y.
\]

or in differential form

\[
dz = f_x \, dx + f_y \, dy.
\]

These types of formula are useful when we want to approximate the change in output given that we know the approximate changes of our input. In particular this can be used for error tolerance but we can also use this to give approximate values for the function near a point that we can easily evaluate the function.

Tangent planes are trying to mimic the function so that it agrees locally with the function in both the value of the function and the first order partial derivatives of the function. We can also try to find a function that matches the value, the first order partial derivatives and the second order partial derivatives. These are done by using the Taylor polynomials. The second order Taylor polynomials are shown below (where the function and derivatives are all evaluated at the point \( (x_0, y_0) \)):

\[
z = f + f_x(x-x_0) + f_y(y-y_0) + \frac{1}{2}f_{xx}(x-x_0)^2 + f_{xy}(x-x_0)(y-y_0) + \frac{1}{2}f_{yy}(y-y_0)^2.
\]

Optimization

The goal of optimization is to maximize or minimize a function. There are two types of maximums. A global maximum is a point where the function evaluated at that point is at least as large or larger than the function evaluated anywhere else. A local maximum is a point where the function evaluated at that point is at least as large or larger than the function evaluated at points nearby. Similar definitions apply for minimums.

The nice fact is that optimization works similarly to single variable calculus, i.e., we generally look for critical points and then apply a test of some sort. One nice fact that still holds is that if a function is continuous on a closed and bounded set then the function must achieve a maximum and a minimum value on that set.

At a maximum (similarly a minimum) we cannot get bigger, so the gradient should not be nonzero (otherwise moving in the direction of the gradient allows us to increase and moving in the direction opposite the gradient allows us to decrease). So we have that maximums and minimum will occur at critical points, these include:

- Where \( \nabla f = 0 \) (i.e., critical points).
- Where \( \nabla f \) is undefined.
- At boundary.

For a function we find critical points by looking at \( \nabla f \) and where it is \( 0 \), equivalently where the partial derivatives are \( 0 \). Once we have the critical points the next step is to determine what type of critical point it is. In single variable calculus we had the first and second derivative tests; in multi-variable calculus we have the Second Partial Test. This is done by noting that at a critical point the partial derivatives are equal to zero and so nearby using the second order Taylor polynomial we have

\[
f(x, y) \approx f(x_0, y_0) + \frac{1}{2} [\Delta x \, \Delta y] \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} [\Delta x \, \Delta y].
\]

We can use properties of \( 2 \times 2 \) matrices (i.e., eigenvalues) to link the local behavior to the determinant of this matrix, i.e.,

\[
D = f_{xx} f_{yy} - (f_{xy})^2.
\]

We have the following possibilities

- If \( D > 0 \) and \( f_{xx} < 0 \) (or \( f_{yy} < 0 \)) then it is a local maximum.
- If \( D > 0 \) and \( f_{xx} > 0 \) (or \( f_{yy} > 0 \)) then it is a local minimum.
- If $D < 0$ then it is a saddle, neither max or min.
- If $D = 0$ the test is inconclusive.

When finding the maximum and/or minimum on a closed and bounded set we know that it must exist, so we find all the places where it could exist and then test each point. In short we do the following:

1. Find all critical points in the interior using $\nabla f$.
2. Find all critical points on boundary (reduce dimensions down).
3. Plug in list of critical points into function.
4. Largest number on list is max; smallest is min.

**Lagrange multipliers**

The technique of Lagrange multipliers is used to solve optimization problems with a constraint. These are usually easy to identify, i.e., there will be two functions one that is being maximized and the other that is a constraint on the variables (i.e., “given that” or “such that”).

Essentially what it will boil down to is that if the gradients of the function we are optimizing and the function that is our constraint are not parallel then by slightly perturbing along our constraint we can increase or decrease our value. So we can conclude that if we are at a point where we are maximizing or minimizing the two gradient vectors must be parallel (this includes the possibility that one of them is 0).

So if maximizing the function $f(x,y)$ given that $g(x,y) = k$ then the method of Lagrange multipliers reduces down to solving the following equations:

$$\nabla f(x,y) = \lambda \nabla g(x,y) \quad \text{and} \quad g(x,y) = k.$$ 

This leads to a large system of nonlinear equations (i.e., one for each partial derivative and one for the constraint).

When in doubt on how to solve such a system the following technique tends to work: Solve for $\lambda$ and set the various terms equal to lambda equal to one another. This gives another relationship between $x$ and $y$ that can be used with $g(x,y) = k$ to solve for the possible points yielding a maximum and/or minimum. Once we have these points we plug into the function, the largest value is the maximum and the smallest value is the minimum.

**Quiz 8 problems**

1. Find the second order Taylor polynomial for $f(x, y) = e^{x+y^2} + x \sin y$ at the point $(0,0)$.
2. A machine is being set up to make cones to be used for storage. The volume of a cone is $V = \frac{1}{3} \pi r^2 h$ where $r$ is a radius and $h$ is a height. The machine ideally makes the cones with $r = 3$ and $h = 5$ for a total volume of $15\pi$ units however there tend to be slight imperfections. Given that the volume of a cone must be within $\pi$ units (i.e., $\Delta V \approx \pm \pi$) and that the height of the cone as the machine is set up has $\Delta h \approx \pm 0.2$, then estimate the tolerance that we need to have for the radius (i.e., find $\Delta r$).
3. Find and classify the critical points for

$$f(x, y) = x^3 - 8xy + 2y^2 - 3x + 4y - 23.$$ 

4. Find the critical points for

$$g(x, y) = 2x^3 - 2x^2 y + 6xy + y^2 - x^2 + 137$$

5. Let $h(x, y) = y^2 \sin x - x$. Verify that $(0,1)$ is a critical point and classify the point as a maximum, minimum or saddle.
6. The Nook corporation (maker of the finest Yops) has recently merged with the Jibboo conglomerate (maker of the finest Zans). Currently Yops sell for three dollars each and Zans sell for nine dollars each. By combining their production the new company enjoys economy of scope and is now able to produce $y$ Yops and $z$ Zans at a cost of $10 + \frac{1}{2} y^2 + \frac{1}{2} z^2 - yz$ dollars. Determine how many Yops and Zans respectively should be made in order to maximize profit. Also, verify that your answer is a maximum by using the second derivative test.

7. Use the technique of Lagrange multipliers to find the maximum value of $f(x,y) = xy + y$ given that $9x^2 + 10y^4 = 9$.

8. You have recently been hired to paint three large non-overlapping dots (a red dot, a blue dot, and a green dot) on the side of a building. They have left the design of the dots to you, the only instruction they gave is that the total area must be $200\pi$ cubic units. The contract allows you to charge $3\pi$ thousand dollars for painting a red circle of radius $r$, while for a blue circle you charge $4\pi$ thousand and for a green circle you charge $5\pi$ thousand. What radii should you choose for the various circles to maximize how much you charge?