Desingularization of an immersed self-shrinker.

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Examples of gluing constructions

Summary of a classical construction
  Minimal surfaces
  One of the difficulties

An immersed configuration
  The problem
  The fix
Examples of Gluing Constructions
Desingularizations

To desingularize: to transform an immersed surface into an embedded one
Scherk surfaces are **minimal** surfaces:

http://www.indiana.edu/~minimal/archive/Classical/Classical/ShearScherk-anim/web/qt.mov

There is a **one-parameter family** of them:

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Building blocks: Scherk minimal surfaces

Scherk surfaces are minimal surfaces:

There is a one-parameter family of them:

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They enjoy symmetries.
The wings tend to half-planes exponentially fast.
Minimal surfaces, $\tilde{H} = 0$

[Kapouleas, 1997]

Self-translating surfaces: $\tilde{H} + \tilde{e}_z \cdot \tilde{\nu} = 0$

[Nguyen, 2012]

[Traizet, 1996]

[Dávila-Del Pino-N., almost done]
Self-shrinking surfaces: \( \tilde{H} - \frac{1}{2} \tilde{X} \cdot \tilde{\nu} = 0 \)

[Kapouleas–Kleene–Møller], [N.]

The non-compact surface is asymptotic to a cone at infinity.
Common features of the examples

- the equation is of the form

\[ \tilde{H} + \text{stuff} = 0. \] (1)

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  \[ X = \frac{1}{\tau} \tilde{X}, \quad H = \tau \tilde{H} \]
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  \]

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Equation (1) becomes

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Equation (1) becomes

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- it is a desingularization of two or more intersecting surfaces.
Immersed self-shrinking surfaces,

[Drugan-Kleene, preprint]
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Desingularizing immersed surfaces? Same story?
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Desingularizing immersed surfaces? Same story? Yes and no.
Summary of a classical construction: minimal surfaces
Sketch of the proof: minimal surfaces

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   - wrap a Scherk surface around a large circle

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2. Look at **graphs of functions** over this approximate surface:
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   ▶ solve \( L(f) := \Delta f + |A|^2 f = E \) on each of the 5 pieces
   ▶ finish with a fixed point theorem
Linear operator on the Scherk surfaces

\[ L(f) = \Delta f + |A|^2 f \]

is associated to normal perturbations of the mean curvature. The mean curvature is invariant under translations.
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- **Partial solution:** We can impose one symmetry: \( \vec{e}_z - \nu. \)
- **But, what do we do with** \( \vec{e}_x \cdot \nu \) and \( \vec{e}_y \cdot \nu ? \) Define

\[ z_1 = \vec{e}_x \cdot \nu, \quad z_2 = \vec{e}_y \cdot \nu \]
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**Lemma (A priori estimates)**

Let $0 < \gamma < 1$ and

1. $E : \Sigma \rightarrow \mathbb{R}$ with $\|e^{\gamma s} E\|_{\infty} \leq C$

2. $f$ be a bounded solution to $\Delta f + |A|^2 f = E$ in $\Sigma$
Linear operator on the Scherk surfaces: a priori estimates

We want exponential decay for $f$. We need a priori estimates for the linear problem on $\Sigma$.

**Lemma (A priori estimates)**

*Let $0 < \gamma < 1$ and*

- $E : \Sigma \to \mathbb{R}$ with $\|e^{\gamma s} E\|_\infty \leq C$
- $\eta_0 : \Sigma \to \mathbb{R}$ a cut-off function
- $f$ be a bounded solution to
  \[ \Delta f + |A|^2 f = E \text{ in } \Sigma \]
  such that $\int_{\Sigma} f \eta_0 z_i = 0$, $i = 1, 2$
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then

$$\|f\|_\infty \leq C\|e^{\gamma s} E\|_\infty,$$

$$|\nabla f| \leq C e^{-\gamma s}(\|e^{\gamma s} E\|_\infty),$$
Linear operator on the Scherk surfaces

- \( z_1 = \vec{e}_x \cdot \nu \), \( z_2 = \vec{e}_y \cdot \nu \)
- \( R > R_1 \) with \( R_1 \) large

Lemma (Existence)

Given \( E : \Sigma \to \mathbb{R} \), there is \( f \) such that

\[
\Delta f + |A|^2 f = E \text{ in } \Sigma_R
\]

\[
f = 0 \text{ on } \partial \Sigma_R
\]

\( \Sigma \) is the entire Scherk surface; \( \Sigma_R \) is \( \Sigma \) truncated at \( s = R \).
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Lemma (Existence)

Given \( E : \Sigma \rightarrow \mathbb{R} \) with \( \| e^{\gamma s} E \|_\infty \leq C \), there are constants \( c_1 \) and \( c_2 \), there is \( f \) such that

\[
\Delta f + |A|^2 f = E + c_1 \eta_0 z_1 + c_2 \eta_0 z_2 \text{ in } \Sigma_R \\
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Given $E : \Sigma \rightarrow \mathbb{R}$ with $\| e^{\gamma s} E \|_\infty \leq C$, there are constants $c_1$ and $c_2$, there is a unique $f$ such that

$$\int_{\Sigma_R} f \eta_0 z_i = 0, \quad i = 1, 2$$

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Linear operator on the Scherk surfaces

- \( z_1 = \varepsilon_x \cdot \nu \), \( z_2 = \varepsilon_y \cdot \nu \)
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\]

\[
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\]

Moreover, \( |c_1|, |c_2|, \| f \|_{\infty} \leq C \| e^{\gamma s} E \|_{\infty} \)

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Achieving exponential decay

Solve on $\Sigma_R$, then let $R \rightarrow \infty$. We get a solution $f$ on $\Sigma$. 

\[ |\nabla f| \sim e^{-\gamma s} \text{ and } |f| \leq C \]

On each wing, $\lim_{s \to \infty} f = \text{constant} L$

$f + c_1' z_1 + c_2' z_2$ has zero limit on two adjacent wings.

$f + c_1' z_1 + c_2' z_2 + \tilde{c}_1 \zeta_1 z_1 + \tilde{c}_2 \zeta_2 z_1$ has zero limit on all wings.
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- On each wing, $\lim_{s \to \infty} f = \text{constant } L_i$
- $f + c'_1 z_1 + c'_2 z_2$ has zero limit on two adjacent wings.
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- $|\nabla f| \sim e^{-\gamma s}$ and $|f| \leq C$
- On each wing, $\lim_{s \to \infty} f = \text{constant } L_i$
- $f + c'_1 z_1 + c'_2 z_2$ has zero limit on two adjacent wings.
- $f + c'_1 z_1 + c'_2 z_2 + \tilde{c}_1 \zeta_1 z_1 + \tilde{c}_2 \zeta_2 z_1$ has zero limit on all wings.

$\zeta_1$ and $\zeta_2$ are smooth cut-off functions on separate wings. $\zeta_i = 1$ for $s$ large, $i = 1, 2$
Linear operator on the Scherk surfaces

Lemma
Given $E : \Sigma \rightarrow \mathbb{R}$ with $\| e^{\gamma s} E \|_{\infty} \leq C$, there are constants $c_1, c_2, \tilde{c}_1,$ and $\tilde{c}_2$, there is a function $\tilde{f}$ such that

$$L \tilde{f} = E + \tilde{c}_1 L(\zeta_1 z_1) + \tilde{c}_2 L(\zeta_2 z_1) + c_1 \eta_0 z_1 + c_2 \eta_0 z_2 \text{ in } \Sigma$$

$$\lim_{s \rightarrow \infty} \tilde{f} = 0 \text{ on each wing}$$

$$|\nabla \tilde{f}| \leq C e^{-\gamma s} \| e^{\gamma s} E \|_{\infty}$$
Dislocations of the Scherk surface

We have to compensate for the four extra terms with dislocations of the Scherk surface.

\[ \hat{L} \tilde{f} = E + \tilde{c}_1 L(\zeta_1 z_1) + \tilde{c}_2 L(\zeta_2 z_1) + c_1 \eta_0 z_1 + c_2 \eta_0 z_2 \]
Dislocations of the Scherk surface

\[
\tilde{Lf} = E + \tilde{c}_1 L(\zeta_1 z_1) + \tilde{c}_2 L(\zeta_2 z_1) + c_1 \eta_0 z_1 + c_2 \eta_0 z_2 \\
+ d_1 L(z_{trans}^1) + d_2 L(z_{trans}^2)
\]
Dislocations of the Scherk surface

\[ \tilde{L}f = E + \tilde{c}_1 L(\zeta_1 z_1) + \tilde{c}_2 L(\zeta_2 z_1) + c_1 \eta_0 z_1 + c_2 \eta_0 z_2 \]
\[ + d_1 L(z^1_{trans}) + d_2 L(z^2_{trans}) + \theta_1 L(z^1_{rot}) + \theta_2 L(z^2_{rot}) \]
If you remember one thing...

The linear problem on the Scherk surfaces is well-studied [Kapouleas, 1997]. We can use it! For a construction to work, one “just” needs

- flexibility of the initial setting.
- solve the associated linear problem on all the non-Scherk pieces.

Setting up the fixed point theorem at the end is also well-documented.
An immersed configuration
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The flexibility is still there

First unwind the bowtie to a segment $[s_-, s_+]$, $s_- < 0 < s_+$. 

\[
\mathcal{L} f_d = 0, \\
 f_d(0) = 1, \quad \dot{f}_d(0) = 0
\]

Record the Dirichlet and Neumann data at the ends
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\[
\mathcal{L} f_n = 0, \\
Q f_n(0) = 0, \quad \dot{f}_n(0) = 1
\]

Record the Dirichlet and Neumann data at the ends
The flexibility is still there

We can impose Dirichlet and Neumann defects

\[ \text{iff } \det\left(\begin{array}{cc} d_2 & d_1 \\ n_2 & n_1 \end{array}\right) \neq 0 \]

\[ \text{iff there are no nontrivial solutions to } Lf = 0 \text{ globally.} \]
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We can impose Dirichlet and Neumann defects

iff det \[
\begin{pmatrix}
  d_2 - d_1 & d_4 - d_3 \\
  n_2 - n_1 & n_4 - n_3
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- iff \( \det \begin{pmatrix} d_2 - d_1 & d_4 - d_3 \\ n_2 - n_1 & n_4 - n_3 \end{pmatrix} \neq 0 \)
- iff there are no nontrivial solution to \( \mathcal{L}f = 0 \) globally.
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- One can solve \((\Delta + |A|^2)f = E\) on \(\Sigma_R\) with

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\|f\|_{C^2,\alpha} \sim e^R \|E\|_{C^0,\alpha}
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  \[
  \|f\|_{C^{2,\alpha}} \sim e^R \|E\|_{C^{0,\alpha}}
  \]
- We are working on a Dirichlet to Neumann map to finish the construction.
The bowtie exists!
Immediate work and questions

Work in progress

- The Dirichlet to Neumann map is invertible for $\tau$ small.
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Questions for you

- Extensions of the work? Related problems? Applications?
Thank you for your attention!