Desingularization of an immersed self-shrinker.

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Examples of gluing constructions

Summary of a classical construction
  Minimal surfaces
  One of the difficulties

An immersed configuration
  The problem
  The fix
Examples of Gluing Constructions
Desingularizations

To desingularize: to transform an immersed surface into an embedded one
Building blocks: Scherk minimal surfaces

Scherk surfaces are \textit{minimal} surfaces:

![Scherk surface diagrams](http://www.indiana.edu/~minimal/archive/Classical/Classical/ShearScherk-anim/web/qt.mov)

There is a \textit{one-parameter family} of them:

http://www.indiana.edu/~minimal/archive/Classical/Classical/ShearScherk-anim/web/qt.mov

They enjoy \textit{symmetries}.

The wings tend to half-planes \textit{exponentially} fast.
Minimal surfaces, $\tilde{H} = 0$

[Kapouleas, 1997]

Self-translating surfaces: $\tilde{H} + \tilde{e}_z \cdot \tilde{\nu} = 0$

[Nguyen, 2012]

[Traizet, 1996]

[Dávila-Del Pino-N., almost done]
Self-shrinking surfaces: \( \tilde{H} - \frac{1}{2} \tilde{X} \cdot \tilde{\nu} = 0 \)

[Kapouleas–Kleene–Møller], [N.]

The non-compact surface is asymptotic to a cone at infinity.
Common features of the examples

▶ the equation is of the form

\[ \tilde{H} + \text{stuff} = 0. \]  \hspace{1cm} (1)

▶ the stuff above scales properly:

\[ X = \frac{1}{\tau} \tilde{X}, \quad H = \tau \tilde{H} \]

Equation (1) becomes

\[ H + \tau \text{ stuff} = 0 \]

▶ it is a desingularization of two or more intersecting surfaces.
Immersed self-shrinking surfaces,

[Drugan-Kleene, preprint]

Desingularizing immersed surfaces? Same story? Yes and no.
Summary of a classical construction: minimal surfaces
Sketch of the proof: minimal surfaces

1. Construct an approximate solution:
   - wrap a Scherk surface around a large circle
   - scale by $\tau$, a small constant
     (or rather, scale everything else by $\frac{1}{\tau}$).

2. Look at graphs of functions over this approximate surface:
   - position $X \rightarrow X + f\nu$
   - mean curvature $H \rightarrow H + \Delta f + |A|^2 f + \text{Quadratic}$
   - solve $L(f) := \Delta f + |A|^2 f = E$ on each of the 5 pieces
   - finish with a fixed point theorem
Linear operator on the Scherk surfaces

\[ L(f) = \Delta f + |A|^2 f \]

is associated to normal perturbations of the mean curvature. The mean curvature is invariant under translations.

- **Problem:** \( \vec{e}_x \cdot \nu, \vec{e}_y \cdot \nu, \) and \( \vec{e}_z \cdot \nu \) are in the kernel of \( L. \)
- **Partial solution:** We can impose one symmetry: \( \vec{e}_z \rightarrow \nu. \)
- **But, what do we do with** \( \vec{e}_x \cdot \nu \) and \( \vec{e}_y \cdot \nu \)? Define

\[ z_1 = \vec{e}_x \cdot \nu, \quad z_2 = \vec{e}_y \cdot \nu \]
We want exponential decay for $f$. We need a priori estimates for the linear problem on $\Sigma$.

**Lemma (A priori estimates)**

Let $0 < \gamma < 1$ and

- $E : \Sigma \to \mathbb{R}$ with $\|e^{\gamma s}E\|_{\infty} \leq C$
- $\eta_0 : \Sigma \to \mathbb{R}$ a cut-off function
- $f$ be a bounded solution to
  \[
  \Delta f + |A|^2 f = E \text{ in } \Sigma
  \]
  such that $\int_{\Sigma} f \eta_0 z_i = 0, i = 1, 2$

then

\[
\|f\|_{\infty} \leq C\|e^{\gamma s}E\|_{\infty},
\]
\[
|\nabla f| \leq Ce^{-\gamma s}(\|e^{\gamma s}E\|_{\infty}),
\]
Linear operator on the Scherk surfaces

- $z_1 = \bar{e}_x \cdot \nu$, $z_2 = \bar{e}_y \cdot \nu$
- $R > R_1$ with $R_1$ large
- $0 < \gamma < 1$
- $\eta_0 : \Sigma \to \mathbb{R}$ a cut-off function

Lemma (Existence)

Given $E : \Sigma \to \mathbb{R}$ with $\| e^{\gamma s} E \|_\infty \leq C$, there are constants $c_1$ and $c_2$, there is a unique $f$ such that $\int_{\Sigma_R} f \eta_0 z_i = 0$, $i = 1, 2$

$$\Delta f + |A|^2 f = E + c_1 \eta_0 z_1 + c_2 \eta_0 z_2 \text{ in } \Sigma_R$$
$$f = 0 \text{ on } \partial \Sigma_R$$

Moreover, $|c_1|, |c_2|, \| f \|_\infty \leq C \| e^{\gamma s} E \|_\infty$

$\Sigma$ is the entire Scherk surface; $\Sigma_R$ is $\Sigma$ truncated at $s = R$. 

$\mathbb{R}$
Achieving exponential decay

Solve on $\Sigma_R$, then let $R \to \infty$. We get a solution $f$ on $\Sigma$.

- $|\nabla f| \sim e^{-\gamma s}$ and $|f| \leq C$
- On each wing, $\lim_{s \to \infty} f = \text{constant } L_i$
- $f + c_1' z_1 + c_2' z_2$ has zero limit on two adjacent wings.
- $f + c_1' z_1 + c_2' z_2 + \tilde{c}_1 \zeta_1 z_1 + \tilde{c}_2 \zeta_2 z_1$ has zero limit on all wings.

$\zeta_1$ and $\zeta_2$ are smooth cut-off functions on separate wings.
$\zeta_i = 1$ for $s$ large, $i = 1, 2$
Lemma

Given $E : \Sigma \to \mathbb{R}$ with $\|e^{\gamma s} E\|_\infty \leq C$, there are constants $c_1$, $c_2$, $\tilde{c}_1$, and $\tilde{c}_2$, there is a function $\tilde{f}$ such that

$$L \tilde{f} = E + \tilde{c}_1 L(\zeta_1 z_1) + \tilde{c}_2 L(\zeta_2 z_1) + c_1 \eta_0 z_1 + c_2 \eta_0 z_2 \text{ in } \Sigma$$

$$\lim_{s \to \infty} \tilde{f} = 0 \text{ on each wing}$$

$$|\nabla \tilde{f}| \leq C e^{-\gamma s} \|e^{\gamma s} E\|_\infty$$
Dislocations of the Scherk surface

We have to compensate for the four extra terms with dislocations of the Scherk surface.

\[ \tilde{L} \tilde{f} = E + \tilde{c}_1 L(\zeta_1 z_1) + \tilde{c}_2 L(\zeta_2 z_1) + c_1 \eta_0 z_1 + c_2 \eta_0 z_2 \]
Dislocations of the Scherk surface

\[ \tilde{L}f = E + \tilde{c}_1 L(\zeta_1 z_1) + \tilde{c}_2 L(\zeta_2 z_1) + c_1 \eta_0 z_1 + c_2 \eta_0 z_2 \\
+ d_1 L(z_{trans}^1) + d_2 L(z_{trans}^2) \]
Dislocations of the Scherk surface

\[
\tilde{L}f = E + \tilde{c}_1 L(\zeta_1 z_1) + \tilde{c}_2 L(\zeta_2 z_1) + c_1 \eta_0 z_1 + c_2 \eta_0 z_2 \\
+ d_1 L(z^1_{trans}) + d_2 L(z^2_{trans}) + \theta_1 L(z^1_{rot}) + \theta_2 L(z^2_{rot})
\]
If you remember one thing...

The **linear problem on the Scherk surfaces** is well-studied [Kapouleas, 1997]. We can use it!
For a construction to work, one “just” needs

- **flexibility** of the initial setting.
- solve the associated **linear problem on all the non-Scherk pieces**.

Setting up the **fixed point theorem** at the end is also well-documented.
An immersed configuration
The problem
The flexibility is still there

First unwind the bowtie to a segment \([s_-, s_+]\), \(s_- < 0 < s_+\).

\[
\mathcal{L} f_d = 0, \\
f_d(0) = 1, \quad \dot{f}_d(0) = 0
\]

Record the Dirichlet and Neumann data at the ends

\[
\mathcal{L} f_n = 0, \\
f_n(0) = 0, \quad \dot{f}_n(0) = 1
\]

Record the Dirichlet and Neumann data at the ends
The flexibility is still there

We can impose Dirichlet and Neumann defects

- iff \( \det \begin{pmatrix} d_2 - d_1 & d_4 - d_3 \\ n_2 - n_1 & n_4 - n_3 \end{pmatrix} \neq 0 \)
- iff there are no nontrivial solution to \( \mathcal{L}f = 0 \) globally.
Remarks

- There is no guarantee that the Scherk angle, position, rotation are preserved.
- We are using up all the degrees of freedom.
- One can solve $(\Delta + |A|^2)f = E$ on $\Sigma_R$ with
  \[ \|f\|_{C^2,\alpha} \sim e^R \|E\|_{C^0,\alpha} \]
- We are working on a Dirichlet to Neumann map to finish the construction.
The bowtie exists!
Immediate work and questions

Work in progress

- The Dirichlet to Neumann map is invertible for \( \tau \) small.
- Proof of the existence of the bowtie.

Questions for you

- Extensions of the work? Related problems? Applications?
Thank you for your attention!
Shrinking doughnuts.

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Construction of complete embedded self-similar surfaces under mean curvature flow. Part III. 

Construction de surfaces minimales en recollant des surfaces de Scherk. 

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