Homework 2  math520

Due: Friday Feb. 9

1. Let \( \{h_k\}_{k=1}^{\infty} \) be a sequence of vectors in Hilbert space \( H \).
   
a) If each \( h_k \) is outside the closed span of the other \( h_k \)'s, prove there exists a biorthogonal sequence to \( \{h_k\} \);
   
b) If \( \{h_k\} \) is a spanning set for \( H \) show that there is at most one biorthogonal sequence to \( \{h_k\} \).

2. Show there exist a polynomial \( g \) of degree \( n \) (i.e, in \( P_n \)) for which \( f(0) = \int_0^\infty f(s)g(s)e^{-s} \, ds \) for all \( f \in P_n \). Find the required \( g \) in the case \( n = 2 \).

3. Prove that the null space of a closed operator is closed.

4. Assume \( g \in C[0,1] \). Prove that \( g(x) \frac{d}{dx} \) is closeable.

5. On \( L^2(0,1) \) define \( Au = e^x u - au \), where \( a \) is a real constant. Describe the state of the operator (in words please) for different values of \( a \).

6. A sequence of bounded operators \( T_n : H \to H \) (\( H \) is a Hilbert space) goes to zero uniformly if \( \|T_n\| \to 0 \), strongly if \( \|T_n u\| \to 0 \) for all \( u \in H \), and weakly if \( < T_n u, v > \to 0 \) for all \( u, v \in H \). Show uniform convergence to zero implies strong convergence to zero and this implies weak convergence to zero and give examples to show the reverse implications do not hold.