The small world problem:
Six degrees of graph theory

Ryan Martin

Mathematics Department
Iowa State University
rymartin@iastate.edu
Conestoga Valley ’91

Millersville University/Franklin & Marshall College
Joint Math Colloquium
This talk is based on joint work with:

Alan Frieze
Carnegie Mellon University

Tom Bohman
Carnegie Mellon University

Michael Krivelevich
Tel Aviv University
Bacon game

Graphs

Origins

Erdős number

Many degrees

History

But, seriously...

Connections
Six? degrees of separation

In the Kevin Bacon Game, it is postulated that the center of the Hollywood universe is

Kevin Bacon.
Six? degrees of separation

In the Kevin Bacon Game, it is postulated that the center of the Hollywood universe is
Six? degrees of separation

In the Kevin Bacon Game, it is postulated that the center of the Hollywood universe is

Kevin Bacon.

This is false. It is Dennis Hopper.
Six? degrees of separation

In the Kevin Bacon Game, it is postulated that the center of the Hollywood universe is

Kevin Bacon.

This is false. It is Dennis Hopper.
In the Kevin Bacon Game, it is postulated that the center of the Hollywood universe is

Kevin Bacon.

We link two actors together if they appeared together in the same movie.
Six? degrees of separation

In the Kevin Bacon Game, it is postulated that the center of the Hollywood universe is

Kevin Bacon.

We link two actors together if they appeared together in the same movie.

(They must be together on a cast list at the IMDb.)
Bacon number

The actor’s Bacon number is the fewest number of steps it takes to connect that actor to
Bacon number

The actor’s # is the fewest number of steps it takes to connect that actor to Kevin Bacon.
Bacon number

The actor’s Bacon number is the fewest number of steps it takes to connect that actor to
An actor can have infinite Kevin Bacon number.
Bacon number

An actor can have infinite $\#$. (For example, an actor who appeared alone in only one film.)
Example: Kevin Costner

Kevin Costner is linked to Kevin Bacon because both appeared in

Example: Kevin Costner is linked to because both appeared in Oliver Stone's JFK.
Example: Kevin Costner

Kevin Costner is linked to Kevin Bacon because both appeared in


So, Kevin Costner’s Kevin Bacon number is

???
Example: Kevin Costner

Kevin Costner is linked to Kevin Bacon because both appeared in


So, Kevin Costner’s Kevin Bacon number is

1
Example: Kevin Costner

Kevin Costner is linked to Kevin Bacon because both appeared in *JFK* (1991).

So, Kevin Costner’s # is 1.
Illustration: Kevin Costner
Illustration: Kevin Costner
Illustration: Kevin Costner
Illustration: Kevin Costner
E.g.: Keanu “Woah!” Reeves

We know that

- appeared with in "Devil's Advocate"
E.g.: Keanu “Woah!” Reeves

We know that

- Keanu Reeves appeared with Al Pacino in *The Devil’s Advocate* (1997)

- appeared with in

[Images of Keanu Reeves and Al Pacino]
E.g.: Keanu “Woah!” Reeves

We know that

- Keanu Reeves appeared with Al Pacino in **The Devil’s Advocate** (1997)

- appeared with in
E.g.: Keanu “Woah!” Reeves

We know that

- Keanu Reeves appeared with Al Pacino in *The Devil’s Advocate* (1997)
- appeared with in *Basquiat*
E.g.: Keanu “Woah!” Reeves

We know that

- Keanu Reeves appeared with Al Pacino in **The Devil’s Advocate** (1997)
- Al Pacino appeared with Christopher Walken in **Gigli** (2003)
- Christopher Walken appeared with Courtney Love in **Basquiat** (1996)
- appeared with in **Trapped**
E.g.: Keanu “Woah!” Reeves

We know that

- Keanu Reeves appeared with Al Pacino in *The Devil’s Advocate* (1997)
- Christopher Walken appeared with Courtney Love in *Basquiat* (1996)
- Courtney Love appeared with Kevin Bacon in *Trapped* (2002)
A chain: Keanu “Neo” Reeves
A chain: *Keanu “Neo” Reeves*
A chain: Keanu “Neo” Reeves
A chain: Keanu “Neo” Reeves
A chain: Keanu “Neo” Reeves
A chain: Keanu “Neo” Reeves
A chain: Keanu “Neo” Reeves
A chain: Keanu “Neo” Reeves
A chain: Keanu “Neo” Reeves
A chain: Keanu "Neo" Reeves
More: Keanu “Constantine” Reeves

But it is also true that

- appeared with ??? in Parenthood.
More: Keanu “Constantine” Reeves

But it is also true that

- appeared with in .
But it is also true that


- appeared with in *My Dog Skip*.

More: Keanu “Constantine” Reeves
But it is also true that


Better: Keanu “Excellent!” Reeves
Better: Keanu “Excellent!” Reeves
Better: Keanu "Excellent!" Reeves
Better: Keanu “Excellent!” Reeves
Better: Keanu “Excellent!” Reeves
Better: Keanu “Excellent!” Reeves
Can we do even better?

has never appeared in a film with .
Can we do even better?

Kevin Bacon has never appeared in a film with Keanu Reeves.

So, is
Can we do even better?

Kevin Bacon has never appeared in a film with

So, Keanu Reeves’ Kevin Bacon number is

???
Can we do even better?

Kevin Bacon has never appeared in a film with

So, Keanu Reeves’ Kevin Bacon number is

2
In sum: Keanu “Bogus” Reeves
In sum: Keanu “Bogus” Reeves
In sum: Keanu “Bogus” Reeves
Experimental data

<table>
<thead>
<tr>
<th># of actors</th>
<th># of actors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1806</td>
</tr>
<tr>
<td>2</td>
<td>145024</td>
</tr>
<tr>
<td>3</td>
<td>395126</td>
</tr>
<tr>
<td>4</td>
<td>95497</td>
</tr>
<tr>
<td>5</td>
<td>7451</td>
</tr>
</tbody>
</table>
Experimental data

<table>
<thead>
<tr>
<th># of actors</th>
<th># of actors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1806</td>
</tr>
<tr>
<td>2</td>
<td>145024</td>
</tr>
<tr>
<td>3</td>
<td>395126</td>
</tr>
<tr>
<td>4</td>
<td>95497</td>
</tr>
<tr>
<td>5</td>
<td>7451</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th># of actors</th>
<th># of actors</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>933</td>
</tr>
<tr>
<td>7</td>
<td>106</td>
</tr>
<tr>
<td>8</td>
<td>13</td>
</tr>
</tbody>
</table>
What about the high numbers?

As we said before, there are actors with infinite
What about the high numbers?

As we said before, there are actors with infinite #.

The actors with large # are obscure and the reason why is fairly obvious.
Kevin Bacon not so special

Most successful actors follow the same pattern:
Kevin Bacon not so special

Most successful actors follow the same pattern:

*For every pair of successful actors, they are connected by a path of length $\leq 5$*
Kevin Bacon not so special

Most successful actors follow the same pattern:

For every pair of successful actors, they are connected by a path of length $\leq 5$

Why?
Ask Christopher Walken

What do you think so far, Mr. Walken?
Ask Christopher Walken

What do you think so far, Mr. Walken?
Ask Christopher Walken

What do you think so far, Mr. Walken?
Ask Christopher Walken

What do you think so far, Mr. Walken?
Our model

We will represent actors by VERTICES
Our model

We will represent actors by VERTICES
Our model

We will represent actors by VERTICES

and connect them with EDGES
Our model

We will represent actors by VERTICES

and connect them with EDGES

if they appeared in the same film.
Our model

We will represent actors by VERTICES

and connect them with EDGES

if they appeared in the same film.

This is a GRAPH.
Model parameters

- There are \( n \) actors.
Model parameters

- There are $n$ actors.
- Fix a constant $d$. 
Model parameters

- There are $n$ actors.
- Fix a constant $d$.
- We will begin with an arbitrary graph $H$ such that . . .
Model parameters

- There are $n$ actors.
- Fix a constant $d$.
- We will begin with an arbitrary graph $H$ such that . . .
- . . . in $H$, each actor is connected to at least $dn$ other actors.
Model parameters

- There are $n$ actors.
- Fix a constant $d$.
- We will begin with an arbitrary graph $H$ such that . . .
- . . . in $H$, each actor is connected to at least $dn$ other actors.
- The constant $d$ can be extremely tiny:
Model parameters

- There are $n$ actors.
- Fix a constant $d$.
- We will begin with an **arbitrary** graph $H$ such that . . .
- . . . in $H$, each actor is connected to at least $dn$ other actors.
- The constant $d$ can be extremely tiny:
  
  0.1
Model parameters

- There are $n$ actors.
- Fix a constant $d$.
- We will begin with an arbitrary graph $H$ such that . . .
- . . . in $H$, each actor is connected to at least $dn$ other actors.
- The constant $d$ can be extremely tiny:
  
  $0.1, \ 0.01$
Model parameters

- There are $n$ actors.
- Fix a constant $d$.
- We will begin with an arbitrary graph $H$ such that . . .
- . . . in $H$, each actor is connected to at least $dn$ other actors.
- The constant $d$ can be extremely tiny:
  
  $0.1, \quad 0.01, \quad 10^{-10^{100}}$

It just needs to be independent of $n$. 
Random casting

We add $f(n)$ random casting connections.
Random casting

We add $f(n)$ random casting connections.

What does RANDOM mean?
Random edges

Let $N$ be the number of pairs with no connection between them (NON-EDGES). We can create $m$ new random edges in the following way:
Random edges

Let $N$ be the number of pairs with no connection between them (NON-EDGES). We can create $m$ new random edges in the following way:

- Connect a previously unconnected pair, independently, with probability $m/N$. 
Random edges

Let $N$ be the number of pairs with no connection between them (NON-EDGES). We can create $m$ new random edges in the following way:

- Connect a previously unconnected pair, independently, with probability $m/N$ (COIN FLIPS).
Random edges

Let $N$ be the number of pairs with no connection between them (NON-EDGES). We can create $m$ new random edges in the following way:

- Connect a previously unconnected pair, independently, with probability $m/N$ (COIN FLIPS).
- The average number of new connections is $m$. 
Let $N$ be the number of pairs with no connection between them (NON-EDGES). We can create $m$ new random edges in the following way:

- Connect a previously unconnected pair, independently, with probability $m/N$ (COIN FLIPS).
- The average number of new connections is $m$.

The question:
Random edges

Let $N$ be the number of pairs with no connection between them (NON-EDGES). We can create $m$ new random edges in the following way:

- Connect a previously unconnected pair, independently, with probability $m/N$ (COIN FLIPS).
- The average number of new connections is $m$.

The question:

What is the longest distance between any pair of actors?
Let $N$ be the number of pairs with no connection between them (NON-EDGES). We can create $m$ new random edges in the following way:

- Connect a previously unconnected pair, independently, with probability $m/N$ (COIN FLIPS).
- The average number of new connections is $m$.

The question:

*What is the longest distance between any pair of actors (DIAMETER)?*
Proposition

**Proposition.** If \( f(n) \to \infty \) as \( n \to \infty \), then
Proposition

**Proposition.** If \( f(n) \to \infty \) as \( n \to \infty \), then

\[
\Pr(\text{diam} \leq \ ) \to 1
\]
Proposition

**Proposition.** If \( f(n) \to \infty \) as \( n \to \infty \), then

\[
\text{Pr}(\text{diam} \leq 7) \to 1
\]
Proposition

**Proposition.** If $f(n) \to \infty$ as $n \to \infty$, then

$$\Pr(diam \leq 7) \to 1$$

Important points:
**Proposition**

**Proposition.** If \( f(n) \to \infty \) as \( n \to \infty \), then

\[
\Pr(\text{diam} \leq 7) \to 1
\]

Important points:

- Recall \( f(n) \) is the number of random connections.
\textbf{Proposition}

\textbf{Proposition.} If \( f(n) \to \infty \) as \( n \to \infty \), then
\[
\Pr(\text{diam} \leq 7) \to 1
\]

Important points:

- Recall \( f(n) \) is the number of random connections.
- “7” doesn’t depend on \( d \) at all.
Proposition

**Proposition.** If $f(n) \to \infty$ as $n \to \infty$, then

$$\Pr(\text{diam} \leq 7) \to 1$$

Important points:

- Recall $f(n)$ is the number of random connections.
- “7” doesn’t depend on $d$ at all.
- $f(n)$ can be very small:
**Proposition**

**Proposition.** If \( f(n) \to \infty \) as \( n \to \infty \), then

\[
\Pr(\text{diam} \leq 7) \to 1
\]

Important points:

- Recall \( f(n) \) is the number of random connections.
- “7” doesn’t depend on \( d \) at all.
- \( f(n) \) can be very small:
  \[
  \sqrt{n}
  \]
**Proposition**

**Proposition.** If $f(n) \to \infty$ as $n \to \infty$, then

$$\Pr(\text{diam} \leq 7) \to 1$$

**Important points:**

- Recall $f(n)$ is the number of random connections.
- “7” doesn’t depend on $d$ at all.
- $f(n)$ can be very small:
  - $\sqrt{n}$,
  - $\log n$
**Proposition**

**Proposition.** If $f(n) \to \infty$ as $n \to \infty$, then

$$\text{Pr}(\text{diam} \leq 7) \to 1$$

Important points:

- Recall $f(n)$ is the number of random connections.
- “7” doesn’t depend on $d$ at all.
- $f(n)$ can be very small:
  $$\sqrt{n}, \quad \log n, \quad \sqrt{\log \log \log n}$$
Proof of diameter 7

This is not too hard to prove.
Proof of diameter 7

This is not too hard to prove. Take $H$ and construct $v_1$, 

$\bullet$

$v_1$
Proof of diameter 7

This is not too hard to prove. Take $H$ and construct $v_1, v_2,$
Proof of diameter 7

This is not too hard to prove. Take $H$ and construct $v_1, v_2, v_3$, 

$$v_1, v_2, v_3$$
Proof of diameter 7

This is not too hard to prove. Take $H$ and construct $v_1, v_2, v_3, \ldots$
**Proof of diameter 7**

This is not too hard to prove. Take $H$ and construct $v_1, v_2, v_3, \ldots$ such that $\text{dist}(v_i, v_j) \geq 3$ for $i \neq j$. 

![Diagram showing the construction process]
**Proof of diameter 7**

This is not too hard to prove. Take $H$ and construct $v_1, v_2, v_3, \ldots$ such that $\text{dist}(v_i, v_j) \geq 3$ for $i \neq j$.

So, since they are of distance 3,
Proof of diameter 7

This is not too hard to prove. Take $H$ and construct $v_1, v_2, v_3, \ldots$ such that $\text{dist}(v_i, v_j) \geq 3$ for $i \neq j$.

So, since they are of distance 3, their neighborhoods do not intersect.
Proof of diameter 7

This is not too hard to prove. Take $H$ and construct $v_1, v_2, v_3, \ldots$ such that $\text{dist}(v_i, v_j) \geq 3$ for $i \neq j$.

So, since they are of distance 3, their neighborhoods do not intersect.

The process will end after $\leq \lceil n/(dn + 1) \rceil$ steps.
Proof of diameter 7

This is not too hard to prove. Take $H$ and construct $v_1, v_2, v_3, \ldots$ such that $\text{dist}(v_i, v_j) \geq 3$ for $i \neq j$.

So, since they are of distance 3, their neighborhoods do not intersect.

The process will end after $\leq \lfloor n/(dn + 1) \rfloor$ steps.

The random edges guarantee at least one edge between each pair of the neighborhoods.
The path of length 7

So, every vertex is of distance at most 2 from some $v_i$. 
The path of length 7

So, every vertex is of distance at most 2 from some \( v_i \).
So, every vertex is of distance at most 2 from some $v_i$.

One edge is between each pair of neighborhoods.
The path of length 7

So, every vertex is of distance at most 2 from some $v_i$.

One edge is between each pair of neighborhoods.
The path of length 7

So, every vertex is of distance at most 2 from some \( v_i \).

One edge is between each pair of neighborhoods.

And this gives the path of length 7:
The path of length 7

So, every vertex is of distance at most 2 from some \( v_i \).

One edge is between each pair of neighborhoods.

And this gives the path of length 7:
The path of length 7

So, every vertex is of distance at most 2 from some $v_i$.

One edge is between each pair of neighborhoods.

And this gives the path of length 7:
The path of length 7

So, every vertex is of distance at most 2 from some $v_i$.

One edge is between each pair of neighborhoods.

And this gives the path of length 7:
The path of length 7

So, every vertex is of distance at most 2 from some $v_i$.

One edge is between each pair of neighborhoods.

And this gives the path of length 7:

![Diagram showing the path of length 7 with vertices $u$, $v_i$, $v_j$, and $w$.]
The path of length 7

So, every vertex is of distance at most 2 from some $v_i$.

One edge is between each pair of neighborhoods.

And this gives the path of length 7:
The path of length 7

So, every vertex is of distance at most 2 from some $v_i$.

One edge is between each pair of neighborhoods.

And this gives the path of length 7:
Even better

There’s a better result, which is harder to prove:
Even better

There’s a better result, which is harder to prove:

**Theorem.** [BFKM] If \( f(n) \to \infty \) as \( n \to \infty \), then

\[
\Pr(\text{diam} \leq 5) \to 1
\]
Even better

There’s a better result, which is harder to prove:

**Theorem.** [BFKM] If $f(n) \to \infty$ as $n \to \infty$, then

$$\Pr(\text{diam} \leq 5) \to 1$$

To prove the theorem, you need the **Regularity lemma**.
**Even better**

There’s a better result, which is harder to prove:

**Theorem.** [BFKM] If $f(n) \to \infty$ as $n \to \infty$, then

$$\Pr(\text{diam} \leq 5) \to 1$$

To prove the theorem, you need the **Regularity lemma**.

The Regularity lemma is ’s powerful and complicated graph theoretic tool.
Even better

There’s a better result, which is harder to prove:

**Theorem.** [BFKM] If \( f(n) \to \infty \) as \( n \to \infty \), then

\[
\Pr (\text{diam} \leq 5) \to 1
\]

To prove the theorem, you need the **Regularity lemma**.

The Regularity lemma is Endre Szemerédi’s powerful and complicated graph theoretic tool.
Best possible?

The theorem is “tight”: 
Best possible?

The theorem is “tight”:

If there aren’t an infinite number of edges added, then some $H$’s will be disconnected.
What about closer connections?

- To get $\text{diam} \leq 4$, you need random connections.
- To get $\text{diam} \leq 3$, you need random connections.
- To get $\text{diam} \leq 2$, you need random connections.
What about closer connections?

- To get $\text{diam} \leq 4$, you need $c_1 \log n$ random connections.
- To get $\text{diam} \leq 3$, you need $c_1 \log n$ random connections.
- To get $\text{diam} \leq 2$, you need random connections.
What about closer connections?

- To get $\text{diam} \leq 4$, you need $c_1 \log n$ random connections.
- To get $\text{diam} \leq 3$, you need $c_1 \log n$ random connections.
- To get $\text{diam} \leq 2$, you need $c_2 n \log n$ random connections.
5 degrees!

So, if you have a system, with a linear minimum degree and a little bit of randomness, the diameter will go to 5!
5 degrees!

So, if you have a system, with a linear minimum degree and a little bit of randomness, the diameter will go to 5!

What do you think of that?
5 degrees!

So, if you have a system, with a linear minimum degree and a little bit of randomness, the diameter will go to 5!

What do you think of that?
5 degrees!

So, if you have a system, with a linear minimum degree and a little bit of randomness, the diameter will go to 5!

What do you think of that?

I see.
5 degrees!

So, if you have a system, with a linear minimum degree and a little bit of randomness, the diameter will go to 5!

What do you think of that?

Let’s move to a new problem.
Origins of the problem

The original question posed by Stanley Milgram
The original question posed by Stanley Milgram
The original question posed by Stanley Milgram asked what the **average distance** was among people in a network.
The original question posed by Stanley Milgram asked what the **average distance** was among people in a network.

His methodology was very flawed
**Origins of the problem**

The original question posed by Stanley Milgram asked what the **average distance** was among people in a network.

His methodology was very flawed – he sent out letters from a single source and waited for a return message, many didn’t come back.
Origins of the problem

The original question posed by Stanley Milgram asked what the **average distance** was among people in a network.

His methodology was very flawed – he sent out letters from a single source and waited for a return message, many didn’t come back.

But his assertion that the average distance was around 6 stuck.
Origins of the problem

The original question posed by Stanley Milgram asked what the **average distance** was among people in a network.

His methodology was very flawed – he sent out letters from a single source and waited for a return message, many didn’t come back.

But his assertion that the average distance was around 6 stuck.
Advantages and disadvantages

Our model has some advantages and some disadvantages.

Disadvantages:
Advantages and disadvantages

Our model has some advantages and some disadvantages.

Disadvantages:

• Each person must be connected to at least $dn$ other people.
Advantages and disadvantages

Our model has some advantages and some disadvantages.

Disadvantages:

• Each person must be connected to at least $dn$ other people.
• Not good for modeling the Internet.
Advantages and disadvantages

Our model has some advantages and some disadvantages.

**Disadvantages:**

- Each person must be connected to at least $dn$ other people.
- Not good for modeling the Internet.

**Advantages:**

- Very weak restriction on structure.
- The number of random connections is tiny.
- An upper bound of 5 on the diameter.
- (Since diameter is maximum distance, it is always at most the average distance.)
Advantages and disadvantages

Our model has some advantages and some disadvantages.

**Disadvantages:**
- Each person must be connected to at least $dn$ other people.
- Not good for modeling the Internet.

**Advantages:**
- Very weak restriction on structure.
Advantages and disadvantages

Our model has some advantages and some disadvantages.

Disadvantages:

- Each person must be connected to at least \(dn\) other people.
- Not good for modeling the Internet.

Advantages:

- Very weak restriction on structure.
- The number of random connections is tiny.
Advantages and disadvantages

Our model has some advantages and some disadvantages.

Disadvantages:

- Each person must be connected to at least $dn$ other people.
- Not good for modeling the Internet.

Advantages:

- Very weak restriction on structure.
- The number of random connections is tiny.
- An upper bound of 5 on the diameter.
Advantages and disadvantages

Our model has some advantages and some disadvantages.

**Disadvantages:**
- Each person must be connected to at least $dn$ other people.
- Not good for modeling the Internet.

**Advantages:**
- Very weak restriction on structure.
- The number of random connections is tiny.
- An upper bound of 5 on the diameter.
- (Since diameter is maximum distance, it is always at most the average distance.)
Another model due to Fan Chung and Linyuan Lu
Another model due to Fan Chung and Linyuan Lu eliminates the problems of not being applicable to the Internet.
Another model due to Fan Chung and Linyuan Lu eliminates the problems of not being applicable to the Internet.

However . . .
The Chung-Lu model

Advantages:
Advantages:

- Easy to use, easy to do computations.
The Chung-Lu model

**Advantages:**

- Easy to use, easy to do computations.
- Models the Internet quite well.
The Chung-Lu model

**Advantages:**
- Easy to use, easy to do computations.
- Models the Internet quite well.

**Disadvantages:**
- The number of random connections is huge.
- Weak result: The average distance is $\approx \log_2 n / \log_2 \tilde{d}$, where $\tilde{d}$ relates to the average degree (number of connections).
The Chung-Lu model

Advantages:
- Easy to use, easy to do computations.
- Models the Internet quite well.

Disadvantages:
- The number of random connections is huge.
The Chung-Lu model

Advantages:

• Easy to use, easy to do computations.
• Models the Internet quite well.

Disadvantages:

• The number of random connections is huge.
• Weak result: The average distance is
  \[ \approx \log_2 n / \log_2 \bar{d} \]
The Chung-Lu model

Advantages:
- Easy to use, easy to do computations.
- Models the Internet quite well.

Disadvantages:
- The number of random connections is huge.
- Weak result: The average distance is \( \approx \log_2 n / \log_2 \tilde{d} \), where \( \tilde{d} \) relates to the average degree (number of connections).
Erdős number

One of the most prolific mathematicians of the 20th century was

Paul (Pál) Erdős
March 26, 1913-September 20, 1996
One of the most prolific mathematicians of the 20th century was Paul (Pál) Erdős

March 26, 1913-September 20, 1996
The Erdős number project is concerned with the distance of mathematicians from Paul Erdős.
Erdős number project

The project is concerned with the distance of mathematicians from Paul Erdős.
Erdős number project

The project is concerned with the distance of mathematicians from Paul Erdős.

Two mathematicians are connected if they co-authored a paper together and that paper appears in Mathematical Reviews, accessible by MathSciNet.
Most prolific authors

- : 1401 papers (Erdős number )
Most prolific authors

- Paul Erdős: 1401 papers (Erdős number 0)
- Drumi Bainov: 782 (Erdős number 4)
- Leonard Carlitz: 730 (Erdős number 2)
- Lucien Godeaux: 644 (Erdős number ∞)
- Saharon Shelah: 600 (Erdős number 1)
Most prolific authors

- : 1401 papers (Erdős number )
- Drumi Bainov: 782 (Erdős number )
- Leonard Carlitz: 730 (Erdős number )
Most prolific authors

- •: 1401 papers (Erdős number )
- Drumi Bainov: 782 (Erdős number )
- Leonard Carlitz: 730 (Erdős number )
- Lucien Godeaux: 644 (Erdős number )
Most prolific authors

- Paul Erdős: 1401 papers (Erdős number 0)
- Drumi Bainov: 782 (Erdős number 4)
- Leonard Carlitz: 730 (Erdős number 2)
- Lucien Godeaux: 644 (Erdős number ∞)
- Saharon Shelah: 600 (Erdős number 1)
Most prolific authors

- Paul Erdős: 1401 papers (Erdős number 0)
- Drumi Bainov: 782 (Erdős number 4)
- Leonard Carlitz: 730 (Erdős number 2)
- Lucien Godeaux: 644 (Erdős number ∞)
- Saharon Shelah: 600 (Erdős number 1)
Erdős number statistics

- wrote 1401 papers in Math Reviews.
Erdős number statistics

- wrote 1401 papers in Math Reviews.
- There are 337,000 vertices (authors) in the graph.
Erdős number statistics

- Wrote 1401 papers in Math Reviews.
- There are 337,000 vertices (authors) in the graph.
- There are about 496,000 edges.
Erdős number statistics

- wrote 1401 papers in Math Reviews.
- There are 337,000 vertices (authors) in the graph.
- There are about 496,000 edges.
- Average number of authors per paper: 1.45
Erdős number statistics

- Wrote 1401 papers in Math Reviews.
- There are 337,000 vertices (authors) in the graph.
- There are about 496,000 edges.
- Average number of authors per paper: 1.45
- Average number of papers per author: 6.87
### Experimental data

<table>
<thead>
<tr>
<th>#</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>502</td>
</tr>
<tr>
<td>2</td>
<td>5713</td>
</tr>
<tr>
<td>3</td>
<td>26422</td>
</tr>
<tr>
<td>4</td>
<td>62136</td>
</tr>
<tr>
<td>5</td>
<td>66157</td>
</tr>
</tbody>
</table>
### Experimental data

<table>
<thead>
<tr>
<th>#</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>509</td>
</tr>
<tr>
<td>2</td>
<td>6984</td>
</tr>
<tr>
<td>3</td>
<td>26422</td>
</tr>
<tr>
<td>4</td>
<td>62136</td>
</tr>
<tr>
<td>5</td>
<td>66157</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>#</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>32280</td>
</tr>
<tr>
<td>7</td>
<td>10431</td>
</tr>
<tr>
<td>8</td>
<td>3214</td>
</tr>
<tr>
<td>9</td>
<td>953</td>
</tr>
<tr>
<td>10</td>
<td>262</td>
</tr>
<tr>
<td>11</td>
<td>94</td>
</tr>
</tbody>
</table>

(Most recent data)
# Experimental data

<table>
<thead>
<tr>
<th></th>
<th>#</th>
<th></th>
<th>#</th>
<th></th>
<th>#</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td>6</td>
<td>32280</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>509</td>
<td></td>
<td>7</td>
<td>10431</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6984</td>
<td></td>
<td>8</td>
<td>3214</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>26422</td>
<td></td>
<td>9</td>
<td>953</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>62136</td>
<td></td>
<td>10</td>
<td>262</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>66157</td>
<td></td>
<td>11</td>
<td>94</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>23</td>
<td></td>
<td>13</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>7</td>
<td></td>
<td>15</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Experimental data

<table>
<thead>
<tr>
<th>#</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>509</td>
</tr>
<tr>
<td>2</td>
<td>6984</td>
</tr>
<tr>
<td>3</td>
<td>26422</td>
</tr>
<tr>
<td>4</td>
<td>62136</td>
</tr>
<tr>
<td>5</td>
<td>66157</td>
</tr>
<tr>
<td>6</td>
<td>32280</td>
</tr>
<tr>
<td>7</td>
<td>10431</td>
</tr>
<tr>
<td>8</td>
<td>3214</td>
</tr>
<tr>
<td>9</td>
<td>953</td>
</tr>
<tr>
<td>10</td>
<td>262</td>
</tr>
<tr>
<td>11</td>
<td>94</td>
</tr>
<tr>
<td>12</td>
<td>23</td>
</tr>
<tr>
<td>13</td>
<td>4</td>
</tr>
<tr>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
</tr>
</tbody>
</table>

(R. G. Kamalov)
The unknown mathematician
The unknown mathematician
The unknown mathematician
The unknown mathematician
The unknown mathematician
The unknown mathematician
Applying our model

To have it be very likely that everyone is connected by a path of no more than 5 acquaintances, just arrange a few random meetings.
Applying our model

To have it be very likely that everyone is connected by a path of no more than 5 acquaintances, just arrange a few random meetings.

Think about people at Millersville.
Millersville cliques

Mathematicians
Millersville cliques

Mathematicians

Social Scientists
But for us to get diameter $\leq 5$, we do need each person to know at least $dn$ others before we add few random edges.
More Millersville cliques

But for us to get diameter ≤ 5, we do need each person to know at least $dn$ others before we add few random edges.
Back (to) Bacon

Let us return to the Kevin Bacon question.
Back (to) Bacon

Let us return to the Kevin Bacon question.

We want to find actors with an

- INFINITE

and

- with \# = 8.
Infinite Kevin Bacon number

is someone with infinite

Thomas Alva Edison only appeared in ONE MOVIE (a brief
documentary) and was the only actor.
Infinite Kevin Bacon number

Thomas Alva Edison only appeared in ONE MOVIE (a brief documentary) and was the only actor.

Not soon coming to DVD:
Infinite Kevin Bacon number

is someone with infinite #.

Thomas Alva Edison only appeared in ONE MOVIE (a brief documentary) and was the only actor.

Not soon coming to DVD: MR. EDISON AT WORK IN HIS CHEMICAL LABORATORY (1897).
Kevin Bacon number 7

Joseph Wheeler appeared in two films:

- **General Wheeler and Secretary of War Alger at Camp Wikoff** (1898), a short documentary, in which he appeared with Russell Alexander Alger (Kevin Bacon number 6)
Joseph Wheeler appeared in two films:

- **General Wheeler and Secretary of War Alger at Camp Wikoff** (1898), a short documentary, in which he appeared with Russell Alexander Alger (Kevin Bacon number 6)

- **Surrender of General Toral** (1898), again, a short documentary, with William Rufus Shafter.
Kevin Bacon number 8

William Rufus Shafter also appeared in two films:

- **Surrender of General Toral** (1898) with Joseph Wheeler.
Kevin Bacon number 8

William Rufus Shafter also appeared in two films:

- **Surrender of General Toral** (1898) with Joseph Wheeler.

- **Major General Shafter** (1898) as the only credited cast member.
Conclusion

- Russell Alexander Alger has \( # = 6 \),
Conclusion

- Russell Alexander Alger has $\# = 6$,
- Joseph Wheeler has $\# = 7$, and
Conclusion

- Russell Alexander Alger has $\# = 6$.
- Joseph Wheeler has $\# = 7$, and
- William Rufus Shafter has $\# = 8$. 
The chain

8 William Rufus Shafter was in **SURRENDER OF GENERAL TORAL** (1898) with Joseph Wheeler
The chain

8 William Rufus Shafter was in **Surrender of General Toral** (1898) with Joseph Wheeler

7 Joseph Wheeler was in **General Wheeler and Secretary of War Alger at Camp Wikoff** (1898) with Russell Alexander Alger
The chain

8. William Rufus Shafter was in *Surrender of General Toral* (1898) with Joseph Wheeler

7. Joseph Wheeler was in *General Wheeler and Secretary of War Alger at Camp Wikoff* (1898) with Russell Alexander Alger

6. Russell Alexander Alger was in *President McKinley’s Inspection of Camp Wikoff* (1898) with President William McKinley
The chain

7 Joseph Wheeler was in General Wheeler and Secretary of War Alger at Camp Wikoff (1898) with Russell Alexander Alger

6 Russell Alexander Alger was in President McKinley’s Inspection of Camp Wikoff (1898) with President William McKinley

5 President William McKinley was in President McKinley Taking the Oath (1901) with U. S. Senator Marcus Hanna
The chain

7 Joseph Wheeler was in General Wheeler and Secretary of War Alger at Camp Wikoff (1898) with Russell Alexander Alger

6 Russell Alexander Alger was in President McKinley’s Inspection of Camp Wikoff (1898) with President William McKinley

5 President William McKinley was in President McKinley Taking the Oath (1901) with U. S. Senator Marcus Hanna (R-OH)
The chain

6 Russell Alexander Alger was in President McKinley’s Inspection of Camp Wikoff (1898) with President William McKinley

5 President William McKinley was in President McKinley Taking the Oath (1901) with U. S. Senator Marcus Hanna (R-OH)

4 U. S. Senator Marcus Hanna (R-OH) was in Opening of the Pan-American Exposition Showing Vice President Roosevelt Leading the Procession (1901) with President Theodore Roosevelt
The chain

5 President William McKinley was in President McKinley Taking the Oath (1901) with U. S. Senator Marcus Hanna (R-OH)

4 U. S. Senator Marcus Hanna (R-OH) was in Opening of the Pan-American Exposition Showing Vice President Roosevelt Leading the Procession (1901) with President Theodore Roosevelt

3 President Theodore Roosevelt was in Womanhood, the Glory of the Nation (1917) with Walter McGrail
The chain

4 U. S. Senator Marcus Hanna (R-OH) was in Opening of the Pan-American Exposition Showing Vice President Roosevelt Leading the Procession (1901) with President Theodore Roosevelt

3 President Theodore Roosevelt was in Womanhood, the Glory of the Nation (1917) with Walter McGrail

2 Walter McGrail was in Dick Tracy vs. Crime Inc. (1941) with Wally Rose
3 President Theodore Roosevelt was in *Womanhood, the Glory of the Nation* (1917) with Walter McGrail

2 Walter McGrail was in *Dick Tracy vs. Crime Inc.* (1941) with Wally Rose

1 Wally Rose was in *Murder in the First* (1995) with
The chain

2 Walter McGrail was in **Dick Tracy vs. Crime Inc.** (1941) with Wally Rose

1 Wally Rose was in **Murder in the First** (1995) with
Presidential precedents

William McKinley has a non-cinematic distinction.
Presidential precedents

William McKinley has a non-cinematic distinction.

He was one of four presidents to be assassinated:
Presidential precedents

William McKinley has a non-cinematic distinction.

He was one of four presidents to be assassinated:

McKinley
Sep. 14, 1901
Presidential precedents

William McKinley has a non-cinematic distinction.

He was one of four presidents to be assassinated:

- Lincoln: Apr. 15, 1865
- Garfield: Sep. 19, 1881
- McKinley: Sep. 14, 1901
- Kennedy: Nov. 22, 1963

Dude, what a downer. Let’s at least focus on the wacky one.
Presidential precedents

William McKinley has a non-cinematic distinction.

He was one of four presidents to be assassinated:

Lincoln
Apr. 15, 1865

McKinley
Sep. 14, 1901

Kennedy
Nov. 22, 1963
Presidential precedents

William McKinley has a non-cinematic distinction.

He was one of four presidents to be assassinated:

Lincoln
Apr. 15, 1865

Garfield
Sep. 19, 1881

McKinley
Sep. 14, 1901

Kennedy
Nov. 22, 1963
Presidential precedents

William McKinley has a non-cinematic distinction.

He was one of four presidents to be assassinated:

- Lincoln: Apr. 15, 1865
- Garfield: Sep. 19, 1881
- McKinley: Sep. 14, 1901
- Kennedy: Nov. 22, 1963

Dude, what a downer. Let’s at least focus on the wacky one.
Garfield (not the cat)

- Born in a log cabin in 1831.
- 18 years in the House of Representatives.
- Elected to the Presidency in 1880.
- Shot on July 2, 1881.
- Amateur mathematician.
Garfield (not the cat)

- Born in a log cabin in 1831.
Garfield (not the cat)

- Born in a log cabin in 1831 near Cleveland.
Garfield (not the cat)

- Born in a log cabin in 1831 near Cleveland.
- 18 years in the House of Representatives.
Garfield (not the cat)

- Born in a log cabin in 1831 near Cleveland.
- 18 years in the House of Representatives.
- Elected to the Presidency in 1880.
Garfield (not the cat)

- Born in a log cabin in 1831 near Cleveland.
- 18 years in the House of Representatives.
- Elected to the Presidency in 1880.
- Shot on July 2, 1881.
Garfield (not the cat)

- Born in a log cabin in 1831 near Cleveland.
- 18 years in the House of Representatives.
- Elected to the Presidency in 1880.
- Shot on July 2, 1881, died on September 19, 1881.
Garfield (not the cat)

- Born in a log cabin in 1831 near Cleveland.
- 18 years in the House of Representatives.
- Elected to the Presidency in 1880.
- Shot on July 2, 1881, died on September 19, 1881.
- Amateur mathematician.
Published mathematician

As a Congressman, Garfield got a publication credit:
Published mathematician

As a Congressman, Garfield got a publication credit:

Published mathematician

As a Congressman, Garfield got a publication credit:


Garfield found a proof of the Pythagorean theorem:
Published mathematician

As a Congressman, Garfield got a publication credit:


Garfield found a proof of the Pythagorean theorem:
Garfield’s proof

\[
\text{area of trapezoid} = \text{area of triangle 1} + \text{area of triangle 2} + \text{area of triangle 3}
\]
Garfield’s proof

area of trapezoid = area of triangle 1

+ area of triangle 2

+ area of triangle 3
Garfield’s proof

\[ \text{area of trapezoid} = \text{area of triangle 1} + \text{area of triangle 2} + \text{area of triangle 3} \]
Garfield’s proof

\[
\frac{1}{2}(a + b)(a + b) = \text{area of triangle 1} \\
+ \text{area of triangle 2} \\
+ \text{area of triangle 3}
\]
Garfield's proof

\[
\frac{1}{2}(a + b)(a + b) = \text{area of triangle 1} \\
+ \text{area of triangle 2} \\
+ \text{area of triangle 3}
\]
Garfield’s proof

\[ \frac{1}{2} (a + b)(a + b) = \frac{1}{2} c^2 \]

\[ + \text{ area of triangle 2} \]

\[ + \text{ area of triangle 3} \]
Garfield’s proof

\[
\frac{1}{2}(a + b)(a + b) = \frac{1}{2}c^2
\]

+ area of triangle 2

+ area of triangle 3
Garfield’s proof

\[
\frac{1}{2} (a + b)(a + b) = \frac{1}{2} c^2
\]

\[
+ \frac{1}{2} ab
\]

+ area of triangle 3
Garfield’s proof

\[
\frac{1}{2}(a + b)(a + b) = \frac{1}{2}c^2 \\
+ \frac{1}{2}ab \\
+ \text{area of triangle 3}
\]
Garfield’s proof

\[ \frac{1}{2} (a + b)(a + b) = \frac{1}{2} c^2 \]

\[ + \frac{1}{2} ab \]

\[ + \frac{1}{2} ab \]
Garfield’s proof

\[
\frac{1}{2}(a + b)(a + b) = \frac{1}{2}c^2 \\
+ \frac{1}{2}ab \\
+ \frac{1}{2}ab
\]
Garfield’s proof

\[
\frac{1}{2} (a + b)(a + b) = \frac{1}{2} c^2 + \frac{1}{2} ab + \frac{1}{2} ab
\]
Garfield’s proof

\[(a + b)(a + b) = c^2 + ab + ab\]
Garfield’s proof

\[ a^2 + 2ab + b^2 = c^2 + ab + ab \]
Garfield’s proof

\[ a^2 + b^2 = c^2 \]
Computer networks

Graphs model much more serious stuff.

I.e.,

- computer networks,
**Computer networks**

Graphs model much more serious stuff.

I.e.,

- computer networks,

- shipping routes,
Graphs model much more serious stuff.

I.e.,

- computer networks,
- shipping routes,
- distribution networks.
In networks we are concerned with one particular quantity:

*Network question*
Network question

In networks we are concerned with one particular quantity:

**CONNECTIVITY:** A connected graph is $k$-connected if removing any set of $k - 1$ vertices (and all relevant edges) leaves the graph connected.
Same model

- $n$ computers
Same model

- $n$ computers

- in $H$, each computer is connected to $\geq dn$ others
**Same model**

- $n$ computers
- in $H$, each computer is connected to $\geq dn$ others
- add $f(n)$ random connections
Same model

- $n$ computers
- in $H$, each computer is connected to $\geq dn$ others
- add $f(n)$ random connections

Of course, we want high connectivity with as little randomness as possible.
Connectivity theorem

**Theorem.** [BFKM] Let $k$ be a function of $n$ that is $\ll n$. 
Connectivity theorem

**Theorem.** [BFKM] Let $k$ be a function of $n$ that is $\ll n$. (That is, $k$ grows more slowly than $n$ as $n \to \infty$.)
**Connectivity theorem**

**Theorem.** [BFKM] Let $k$ be a function of $n$ that is $\ll n$. (That is, $k$ grows more slowly than $n$ as $n \to \infty$.) Let $H$ have the property that each vertex is connected to at least $dn$ other vertices.
Connectivity theorem

**Theorem.** [BFKM] Let $k$ be a function of $n$ that is $\ll n$. (That is, $k$ grows more slowly than $n$ as $n \to \infty$.) Let $H$ have the property that each vertex is connected to at least $dn$ other vertices.

- If $f(n) \gg k$, then the graph becomes $k$-connected, with high probability.
**Connectivity theorem**

**Theorem.** [BFKM] Let $k$ be a function of $n$ that is $\ll n$. (That is, $k$ grows more slowly than $n$ as $n \to \infty$.) Let $H$ have the property that each vertex is connected to at least $dn$ other vertices.

- If $f(n) \gg k$, then the graph becomes $k$-connected, with high probability.

- If $d < 1/2$, there is an $H_0$ such that for every $k \ll n$, $f(n) = k - 1$ ensures that the graph fails to be $k$-connected, with high probability.
Bottom line

A way to interpret this theorem is:
A way to interpret this theorem is:

*If you need* \( k \)-connectivity,
Bottom line

A way to interpret this theorem is:

*If you need $k$-connectivity,*

*then you need to add a little more (asymptotically) random edges than $k$.***
A way to interpret this theorem is:

If you need $k$-connectivity,

then you need to add a little more (asymptotically) random edges than $k$.

If fewer than $k$ random edges are added, $k$-connectivity does not necessarily occur.
**Worst case**

What is that $H_0$?

$H_0 =$

Disjoint cliques give the worst case.
We’ve used this model to investigate other properties:

- Hamilton cycle
Other properties

We’ve used this model to investigate other properties:

- Hamilton cycle
- Small cliques as subgraphs
Other properties

We’ve used this model to investigate other properties:

- Hamilton cycle
- Small cliques as subgraphs
- Chromatic number
Other properties

We’ve used this model to investigate other properties:

- Hamilton cycle
- Small cliques as subgraphs
- Chromatic number

Some use the Regularity lemma, some use other techniques.
Bacon and Erdős
Bacon and Erdős
Bacon and Erdős
How are these guys related?
Celebrity nerds

There are some mathematicians and physicists who have Bacon numbers and low Erdős numbers:
Celebrity nerds

There are some mathematicians and physicists who have Bacon numbers and low Erdős numbers:

- Brian Greene (\(d_B = 3\), \(d_E = 2\)) in Frequency (2000).
**Celebrity nerds**

There are some mathematicians and physicists who have Bacon numbers and low Erdős numbers:

- **Brian Greene** ($B=3$, $E=2$) in *Frequency* (2000).

- **Dave Bayer** ($B=3$, $E=2$) in *A Beautiful Mind* (2001).
Nerd celebrities

Danica McKellar
Danica McKellar, math nerd.
Nerd celebrities

Danica McKellar, math nerd. Best known for: The Wonder Years (1988-1993) and
Nerd celebrities

Nerd celebrities

• Danica McKellar was in *Intermission* (2004) with Susan Leslie.
• Susan Leslie was in *Beauty Shop* (2005) with Leeza Gibbons.
Nerd celebrities

- Danica McKellar was in *Intermission* (2004) with Susan Leslie.
Nerd celebrities

- Danica McKellar was in Intermission (2004) with Susan Leslie.
- Susan Leslie was in Beauty Shop (2005) with

[Images of Danica McKellar and Susan Leslie]
Danica’s Math Career

Danica McKellar wrote

Danica’s Math Career

4 Danica McKellar wrote


3 Lincoln Chayes wrote

“No directed fractal percolation in zero area”, which appeared in *The Journal of Statistical Physics*, with Peres and Pemantle.
Danica’s Math Career

3 Lincoln Chayes wrote

“No directed fractal percolation in zero area”,
which appeared in The Journal of Statistical Physics,
with Peres and Pemantle.

2 Robin Pemantle wrote

“Metrics on compositions and coincidences among renewal sequences”,
which appeared in
The IMA Volumes in Mathematics and its Applications,
with Diaconis, Holmes, Lalley and Janson.
2 Robin Pemantle wrote

“Metrics on compositions and coincidences among renewal sequences”,
which appeared in
The IMA Volumes in Mathematics and its Applications,
with Diaconis, Holmes, Lalley and Janson.

1 Svante Janson wrote

“A note on triangle-free graphs”,
which appeared in
The IMA Volumes in Mathematics and its Applications,
with Łuczak, Spencer and Paul Erdős
Danica’s Math Career

1 Svante Janson wrote “A note on triangle-free graphs”, which appeared in The IMA Volumes in Mathematics and its Applications, with Łuczak, Spencer and Paul Erdős.
Erdős, Hollywood Leading Man

Paul Erdős was in *N Is a Number* (1993) with Gene Patterson
Erdős, Hollywood Leading Man

4 Paul Erdős was in *N is a Number* (1993) with Gene Patterson

3 Gene Patterson was in *Box of Moon Light* (1996) with Lisa Blount
Erdős, Hollywood Leading Man

4 Paul Erdős was in *N IS A NUMBER* (1993) with Gene Patterson

3 Gene Patterson was in *Box of Moon Light* (1996) with Lisa Blount

2 Lisa Blount was in *Femme Fatale* (1991) with Colin Firth
Erdős, Hollywood Leading Man

3 Gene Patterson was in **Box of Moon Light** (1996) with Lisa Blount

2 Lisa Blount was in **Femme Fatale** (1991) with Colin Firth

1 Colin Firth was in **Where the Truth Lies** (2005) with Kevin Bacon
Lisa Blount was in *Femme Fatale* (1991) with Colin Firth

Colin Firth was in *Where the Truth Lies* (2005) with Kevin Bacon
And so much more...

This is just a small taste of what graph theory can do.
And so much more...

This is just a small taste of what graph theory can do.
And so much more...

This is just a small taste of what graph theory can do.

The Internet
And so much more...

This is just a small taste of what graph theory can do.

The Internet

(not actual size)
And so much more...

Thank you very much!

The Internet
(not actual size)
More Cowbell!