Recent results on the edit distance of graphs

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Another view of a classical problem

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We want to compute the fewest number of edge-additions plus edge-deletions to apply to $G$ so that the resulting graph $G'$ has no triangles.

Theorem (Mantel, 1907)

If an $n$-vertex graph $G'$ has no triangles, then the number of edges $G'$ has is at most $\left\lfloor \frac{n^2}{4} \right\rfloor$.

This bound is only achieved if $G'$ is complete bipartite.
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So, if $G$ were the complete graph, it would require at least

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edge-deletions.
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edge-deletions.

But for any graph $G$, we can delete at most this many edges and remain triangle-free:

Partition the vertices in half and delete edges inside each part.
Results on triangles

So, the maximum number of changes required to remove triangles from $n$-vertex graph $G$ is

$$\left(\left\lfloor \frac{n}{2} \right\rfloor \right)^2 + \left(\left\lceil \frac{n}{2} \right\rceil \right)^2.$$

This is achieved by $G = K_n$. 
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So, the maximum number of changes required to remove triangles from \( n \)-vertex graph \( G \) is

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\left( \left\lfloor \frac{n}{2} \right\rfloor \right)^2 + \left( \left\lceil \frac{n}{2} \right\rceil \right)^2 \sim \frac{1}{2} \left( \frac{n^2}{2} \right).
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Results on triangles

Instead of triangles, we can generalize the previous analysis to forbid copies of $K_{\ell+1}$, $\ell \geq 2$.

Theorem (Turán, 1941)

*If an n-vertex graph $G'$ has no copy of $K_{\ell+1}$, then the number of edges $G'$ has is at most*

$$\left(1 - \frac{1}{\ell}\right) \frac{n^2}{2}.$$

*This bound is only achieved if $G'$ is complete $\ell$-partite and $\ell \mid n$. 

**Proof:**

[Proof details here]
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$$\sim \frac{n^2}{2\ell} \sim \frac{1}{\ell} \binom{n}{2}.$$

This is achieved by $G = K_n$. 
Given: Labeled graphs $G$ and $G'$, each on $n$ vertices.
Edit Distance between graphs

**Given:** Labeled graphs $G$ and $G'$, each on $n$ vertices.

**Definition**

The **EDIT DISTANCE BETWEEN $G$ AND $G'$**

$$\text{Dist}(G, G') = |E(G) \triangle E(G')|$$

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That is, it is the minimum number of edge-additions plus edge-deletions to transform $G$ into $G'$.
Given: A labeled graph $G$ and a graph property $\mathcal{P}$.

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The **EDIT DISTANCE FROM $G$ TO $\mathcal{P}$**

$$\text{Dist}(G, \mathcal{P}) = \min \{ \text{Dist}(G, G') : G' \in \mathcal{P} \}$$

is the least edit distance of $G$ to a graph in $\mathcal{P}$. 
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Given: A natural number $n$ and a graph property $\mathcal{P}$.
**Extremal Edit Distance**

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<table>
<thead>
<tr>
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Given: A natural number $n$ and a hereditary graph property $\mathcal{H}$.

Definition

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I.e., if \( G \in \mathcal{H} \), then every induced subgraph of \( G \) is in \( \mathcal{H} \).
A hereditary property, $\mathcal{H}$, of graphs is one that holds under the deletion of vertices. I.e., if $G \in \mathcal{H}$, then every induced subgraph of $G$ is in $\mathcal{H}$.

A 5-cycle as a subgraph, but not an induced graph.

A 5-cycle as a subgraph, but not an INDUCED graph.
A **hereditary property**, \( H \), of graphs is one that holds under the deletion of vertices.

I.e., if \( G \in H \), then every induced subgraph of \( G \) is in \( H \).

**Examples:**
- Planar graphs.
- Line graphs of bipartite graphs.
- Chordal graphs: graphs with no chordless cycle of length \( \geq 4 \).
- Perfect graphs: \( \chi = \omega \) for all induced subgraphs.
- Forb(\( H \)): graphs with no induced copy of \( H \).

\(<\text{Forb}(H) is a principal hereditary property.>\)
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  (\( \text{Forb}(H) \) is a **principal hereditary property**.)

For the rest of this talk, all of our hereditary properties are principal; i.e.,

\[
\mathcal{H} = \text{Forb}(H), \text{ for some graph } H.
\]
Recall Dr. Raggi’s dissertation:

Definition

For a configuration, $F$ and a $\{0, 1\}$-matrix $M$, if there is a representation (permutation of the rows and columns) of $F$ that is a submatrix of $M$, then we write $F \prec M$. 

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The edit distance of graphs
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**Definition**

Let $I_r$ be the $r \times r$ identity matrix, $I_r^c$ be its complement and $T_r$ be the $r \times (r + 1)$ “tower matrix”.

\[
P_r(a, b, c) \overset{\text{def}}{=} I_r \times \cdots \times I_r \times I_r^c \times \cdots \times I_r^c \times T_r \times \cdots \times T_r.
\]

\[\begin{array}{cccc}
\text{a times} & \text{b times} & \text{c times}
\end{array}\]
Recall $X(F)$

Let $F$ be a configuration and let $a, b, c \in \mathbb{N}$ have the property that, for all $r \in \mathbb{N}$, $F \not\triangleleft P_r(a, b, c)$.

But for any combination $a', b', c' \in \mathbb{N}$ such that $a' + b' + c' > a + b + c$, then there is an $r$ such that $F \triangleleft P_r(a', b', c')$.

The quantity $a + b + c$ is $X(F)$.

$$P_r(a, b, c) \overset{\text{def}}{=} \underbrace{I_r \times \cdots \times I_r}_{a \text{ times}} \times \underbrace{I_r^c \times \cdots \times I_r^c}_{b \text{ times}} \times \underbrace{T_r \times \cdots \times T_r}_{c \text{ times}}.$$
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A useful parameter

Let $\mathcal{H} = \text{Forb}(H)$ and let $a, c \in \mathbb{N}$ have the property that $V(H)$ cannot be partitioned into a set of $a$ independent sets and $c$ cliques.

But $V(H)$ can be partitioned into ANY combination of $a + c + 1$ independent sets and cliques.
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The quantity $a + c + 1$ is the **binary chromatic number**, $\chi_B(H)$. (It is also called the **colouring number**, $\tau(H)$.)
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**Theorem (Axenovich-Kézdy-M., 2008)**

Let $\mathcal{H} = \text{Forb}(H)$ for some fixed graph $H$ which has coloring number $\tau(H)$,

$$\text{Dist}(n, \text{Forb}(H)) \geq \frac{1}{2(\chi_B(H) - 1)} \binom{n}{2} - o(n^2).$$
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Moreover, this holds with equality if $H$ is self-complementary.
An example: 5-cycle

We will compute $\chi_B(C_5)$, the 5-cycle.
An example: 5-cycle

We will compute $\chi_B(C_5)$, the 5-cycle.

Let us consider all $(a, c)$ such that $a + c = 3$:

The 5-cycle can be partitioned into 3 independent sets and 0 cliques.
An example: 5-cycle

We will compute $\chi_B(C_5)$, the 5-cycle.

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So, $\chi_B(C_5) \leq 3$. 
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So, $\chi_B(C_5) = 3$. 
Edit distance for $C_5$

Since $\chi_B(C_5) = 3$, and $C_5$ is self-complementary, the theorem gives

$$\text{Dist}(n, \text{Forb}(C_5)) = \frac{1}{2(\chi_B(H) - 1)} \binom{n}{2} - o(n^2).$$
Since $\chi_B(C_5) = 3$, and $C_5$ is self-complementary, the theorem gives

$$\text{Dist}(n, \text{Forb}(C_5)) = \frac{1}{2 \cdot 2 \binom{n}{2}} - o(n^2).$$
Edit distance for $C_5$

Since $\chi_B(C_5) = 3$, and $C_5$ is self-complementary, the theorem gives

$$\text{Dist}(n, \text{Forb}(C_5)) = \frac{1}{4} \binom{n}{2} - o(n^2).$$
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Since $\chi_B(C_5) = 3$, and $C_5$ is self-complementary, the theorem gives

$$\text{Dist}(n, \text{Forb}(C_5)) = \frac{1}{4} \binom{n}{2} - o(n^2).$$

Normalize and take the limit:

$$\lim_{n \to \infty} \frac{\text{Dist}(n, \text{Forb}(H))}{\binom{n}{2}} = \frac{1}{4} - o(1).$$
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Normalize and take the limit:

$$\lim_{n \to \infty} \text{Dist}(n, \text{Forb}(H))/\binom{n}{2} = \frac{1}{4}.$$ 

We denote

$$d^*(\mathcal{H}) \overset{\text{def}}{=} \lim_{n \to \infty} \text{Dist}(n, \mathcal{H})/\binom{n}{2}.$$
Edit distance for $C_5$

$$d^*(\mathcal{H}) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{\text{Dist}(n, \mathcal{H})}{\binom{n}{2}}.$$ 

This $d^*(\mathcal{H})$ is the quantity we want to compute.

It’s the proportion of pairs that need to be changed, in the worst case.
Let $G_{n,p}$ denote the Erdős-Rényi random graph:

I.e., there are $n$ vertices and each edge is present, independently, with probability $p$. 

Theorem (Alon-Stav, 2008)
For every hereditary property, $H$, there exists a $p^* = p^*(H) \in [0,1]$ such that 
$$d^*(H) = \lim_{n \to \infty} \frac{\text{Dist}(G_{n,p}, H)}{n^2}.$$ 

Theorem (Balogh-M., 2008) 
For every hereditary property, $H$, and every $p \in [0,1]$, if $g_H(p) = \lim_{n \to \infty} \frac{\text{Dist}(G_{n,p}, H)}{n^2}$ 

then it is also true that $g_H(p) = \lim_{n \to \infty} \max \left\{ \frac{\text{Dist}(G, H)}{n^2} : |V(G)| = n, |E(G)| = p(n^2) \right\} / n^2$. 

Roughly, the hardest density-$p$-graph to edit is the random graph.
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For every hereditary property, $\mathcal{H}$, there exists a $p^* = p^*(\mathcal{H}) \in [0,1]$ such that

$$d^*(\mathcal{H}) = \lim_{n \to \infty} \frac{\text{Dist}(G_{n,p^*}, \mathcal{H})}{\binom{n}{2}}.$$ 

(The expression $\text{Dist}(G_{n,p^*}, \mathcal{H})$ is tightly concentrated about the mean, so the right-hand side is well-defined.)
Understanding $d^*$

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**Theorem (Balogh-M., 2008)**

*For every hereditary property, $\mathcal{H}$, and every $p \in [0, 1]$, if*

$$g_{\mathcal{H}}(p) = \lim_{n \to \infty} \frac{\text{Dist}(G_n, p, \mathcal{H})}{\binom{n}{2}}$$

*then it is also true that*

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Roughly, the hardest density-$p$ graph to edit is the random graph.
Properties of $g_{\text{Forb}}(H)(p)$

Continuous and concave down.

$g_{\text{Forb}}(H)(p) = \frac{1}{2}(\chi_B(H) - 1)$.

Achieves its maximum $(p^*, d^*)$ for some $p^* \in [0, 1]$.

If $H$ is neither complete nor empty, then $g_{\text{Forb}}(H)(0) = g_{\text{Forb}}(H)(1) = 0$.

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Theorem (Balogh-M., 2008)

\[ p^*(\text{Forb}(K_{3,3})) = \sqrt{2} - 1 \quad d^*(\text{Forb}(K_{3,3})) = 3 - 2\sqrt{2}. \]
Example: $C_t$

Theorem (Marchant-Thomason, 2010)

$$g_{\text{Forb}}(C_4)(p) = p(1 - p) \quad d^*_{\text{Forb}}(C_4) = \frac{1}{4}.$$
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**Theorem (Marchant, 2010++)**

$$g_{\text{Forb}}(C_5)(p) = \min \left\{ \frac{p}{2}, \frac{1 - p}{2} \right\} \quad d^*_{\text{Forb}}(C_5) = \frac{1}{4}.$$
Example: $C_t$

**Theorem (M., 2010+)**

$$g_{\text{Forb}(C_6)}(p) = \min \left\{ p(1 - p), \frac{1 - p}{2} \right\} \quad d^*_{\text{Forb}(C_6)} = \frac{1}{4}.$$
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**Theorem (Marchant, 2010++)**

$$g_{\text{Forb}}(C_7)(p) = \min \left\{ \frac{p}{2}, \frac{p(1-p)}{1+p}, \frac{1-p}{3} \right\} \quad d^*_\text{Forb}(C_7) = 3 - 2\sqrt{2}. $$
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**Theorem (M., 2010+)**

$$g_{\text{Forb}}(C_8)(p) = \min \left\{ \frac{p(1 - p)}{1 + p}, \frac{1 - p}{3} \right\} \quad d^*_{\text{Forb}}(C_8) = 3 - 2\sqrt{2}.$$
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$$g_{\text{Forb}}(C_9)(p) = \min \left\{ \frac{p}{2}, \frac{1-p}{4} \right\} \quad d^*_{\text{Forb}}(C_9) = \frac{1}{6}. $$
Example: $C_t$

**Theorem (M., 2010+)**

\[
g_{\text{Forb}(C_{10})}(p) = \min \left\{ \frac{p(1 - p)}{1 + 2p}, \frac{1 - p}{4} \right\} \\
d^*_{\text{Forb}(C_{10})} = \frac{2 - \sqrt{3}}{2}.
\]
Example: $H_9$

**Theorem (M., 2010+)**

$$g_{\text{Forb}}(H_9)(p) = \min \left\{ \frac{p}{3}, \frac{p}{1 + 4p}, \frac{1 - p}{2} \right\}$$

$$d^*_{\text{Forb}}(K_{2,3}) = \frac{7 - \sqrt{17}}{16}.$$
Example: $K_{2,t}$

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**Theorem (M.-McKay, 2010++)**

$$g_{\text{Forb}}(K_{2,3})(p) = \min \left\{ p(1-p), \frac{1-p}{2} \right\}$$

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Example: $K_{2,t}$

**Theorem (M.-McKay, 2010++)**

$$g_{\text{Forb}}(K_{2,4})(p) = \min \left\{ p(1 - p), \frac{1 + 7p}{15}, \frac{1 - p}{3} \right\}$$

$$d^*_{\text{Forb}}(K_{2,4}) = \frac{2}{9}.$$
A CRG from an SRG

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The SRG is the generalized quadrangle \( GQ(2, 2) \). It has the following properties:

- has 15 vertices,
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Upper bounds

Computing upper bounds for $g_{\text{Forb}}(H)(p)$ is easy:

- Begin with a **COLORED REGULARITY GRAPH (CRG)**, or a “recipe” that tells how to partition the vertices of the random graph, $G(n, p)$ and how to add/delete edges to have no induced $H$. 
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- Optimally weight the vertices of the CRG, depending on $p$, to minimize the number of edge-operations.
- This gives an upper bound for $g$. 

![Diagram of a colored regularity graph](image)
Computing lower bounds

Lower bounds continue to be difficult.

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We use a technique we called “localization” which exploits the fact that the optimal weighting of vertices is a quadratic program.

$$
\min \left\{ \bar{x}^T M \bar{x} : \bar{x}^T \bar{1} = 1, \bar{x} \geq \bar{0} \right\}.
$$
Marchant and Thomason show that constructions used to solve the Zarankiewicz problem are necessary for the edit distance function for $K_{3,3}$. This is not true for $K_{2,2}$, $K_{2,3}$ or $K_{2,4}$.
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- With Tracy McKay, we have more results on computing $g_{\text{Forb}}(K_{2,t})(p)$ for $t \geq 5$. The “Zarankiewicz effect” seems to happen for $t \geq 9$.
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- Graph limits, as studied by Borges, Chayes, Elek, Lovász, B. Szegedy, Vesztergombi, et al.
An unusual conjecture

What if the graph $H$ we want to edit away is a random graph, $G(n_0, p_0)$?

Formally,

**Conjecture**

Fix $p_0 \in [0, 1]$ and let $H \sim G(n_0, p_0)$ with $\mathcal{H} = \text{Forb}(H)$. Then, with prob. $\to 1$ as $n_0 \to \infty$,

$$g_{\mathcal{H}}(p) = \min \left\{ \frac{2 \log_2 n_0}{n_0 \log_2 \frac{1}{1-p_0}} \cdot p, \quad \frac{2 \log_2 n_0}{n_0 \log_2 \frac{1}{p_0}} \cdot (1-p) \right\} \pm \mathcal{o}(1).$$

Alon and Stav verified this for $p_0 = \frac{1}{2}$, yielding $p^* = \frac{1}{2}$. I.e., it would be harder to edit $G(n_0, \frac{1}{2})$ out of a $G(n_0, \frac{1}{2})$ than a $G(n_0, \frac{2}{3})$. 
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I.e., it would be harder to edit $G(n_0, 0.25)$ out of a $G \sim G(n, 0.172)$ than a $G \sim G(n, 0.25)$. 
Thanks!

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http://orion.math.iastate.edu/rymartin

My CV (with links to this and previous talks):

http://orion.math.iastate.edu/rymartin/cv/cv.pdf

Contact me:

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