SHOW ALL YOUR WORK to avoid loss of points.

1. Short Answer Questions:
   
a) (1 pt) The equation $x'' + 2x' + x = 0$ describes the motion of a free damped mass-spring system. Decide whether the system is overdamped, critically damped or underdamped.
   
   $\text{roots } r = \frac{-2 \pm \sqrt{4-1}}{2} = -1$ (zero discriminant $\Rightarrow$ critically damped)
   
   b) (4 pt) Find the form of a particular solution $y_p$ (for the method of undetermined coefficients) to the equation $y'' - y = -7e^{-x}$. (Do not find the coefficients!)
   
   \[ m = 1, -1 \Rightarrow e^{-x} \text{ is a sol. So } y_p = x A e^{-x} \]
   
   c) (5 pt) Establish the differential equation and initial condition required to find the amount of salt $A(t)$ in the following problem. (Do NOT solve the equation). A brine with concentration 2 kg/L is being pumped at a rate of 8 L/min into a tank containing originally 100 L of pure water. Suppose the contents of the tank are well kept well stirred and the mixture is pumped out of the tank at the same rate.

   \[ \text{pure H}_2\text{O} \Rightarrow A(0) = 0 \]
   
   \[ \frac{dA}{dt} = \text{rate in} - \text{rate out} \]
   
   \[ \frac{dA}{dt} = \frac{8}{m} \cdot \frac{2 \text{ kg}}{L} - \frac{A(t) \text{ kg}}{100 \text{ L}} \cdot \frac{8 \text{ L}}{m} \]

2. (10pt) Given that the equation $y'' - 8y' + 16y = 50e^{-x}$ has the particular solution $y_p = 2e^{-x}$, find the solution of the associated initial value problem when $y(0) = 0$ and $y'(0) = 10$.

   \[ \text{aux: } Eqn \Rightarrow m^2 - 8m + 16 = 0 \]
   
   \[ (m-4)^2 = 0 \]
   
   \[ m = 4 \text{ repeated} \]
   
   \[ y_c = C_1 e^{4x} + C_2 xe^{4x} \]
   
   \[ \text{Gen. Sol: } y = C_1 e^{4x} + C_2 xe^{4x} + 2e^{-x} \]
   
   \[ y' = 4C_1 e^{4x} + 4C_2 xe^{4x} + 4C_2 e^{4x} - 2e^{-x} \]
   
   \[ \Rightarrow y(0) = C_1 + 2 = 0 \Rightarrow C_1 = -2 \]
   
   \[ y'(0) = 4C_1 + C_2 - 2 = 10 \Rightarrow C_2 = 20 - 4(-2) \]
   
   \[ C_2 = 20 \]

   \[ \Rightarrow y(x) = -2e^{4x} + 20xe^{4x} + 2e^{-x} \]
3. (10pt) Use the method of variation of parameters to find a particular solution \( y_p \) to the equation \( y'' + y = \csc x \)

\[
\text{Aux. Eqn: } m^2 + 1 = 0 \Rightarrow m = \pm i \quad (\alpha = 0, \beta = 1)
\]

\[
y_c = C_1 \cos x + C_2 \sin x \\
\text{W} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1
\]

\[
y_p = u_1 y_1 + u_2 y_2
\]

\[
u_1 = -\int \frac{c_1 \cos x \cdot \sin x}{y_1} \, dx = -\int 1 \, dx = -x
\]

\[
u_2 = \int c_2 \cos x \cdot \cos x \, dx = \int \frac{\cos x}{\sin x} \, dx = \ln |\sin x|
\]

\[
y_p = -x \cos x + \ln |\sin x| \sin x
\]

4. (10pt) Find the general solution of the Cauchy-Euler equation \( 2x^2 y'' - 4xy' - 8y = 0 \).

Sol. of the form \( y = x^m \quad y' = m x^{m-1} \quad y'' = m(m-1) x^{m-2} \)

\[
2 m (m-1) x^m - 4 m x^m - 8 x^m = 0
\]

\[
x^m (2m^2 - 2m - 8) = 0 \quad \Rightarrow \quad m^2 - 3m - 4 = 0
\]

\[
\Rightarrow \ (m + 1)(m - 4) = 0 \quad \Rightarrow \quad m = -1, 4
\]

\[
y = C_1 x^{-1} + C_2 x^4
\]
5. (10pt) A mass weighing 4-lb stretches a spring one foot. At time zero the mass is released from a point 6 inches above equilibrium position, with a downward velocity of 2 ft/s. Determine the equation of motion of the mass, its amplitude and its period.

\[ F = k \cdot s \quad \Rightarrow \quad 4 = k \cdot 1 \quad \Rightarrow \quad k = 4 \]

\[ W = mg \quad \Rightarrow \quad m = \frac{4}{32} \quad \Rightarrow \quad m = \frac{1}{8} \]

Equation: \[ x'' + \frac{k}{m} x = 0 \quad \Rightarrow \quad x'' + 4.8x = 0 \quad ; \quad x(0) = -\frac{1}{2} , \quad x'(0) = 2 \]

Aux Eqn: \[ m^2 + 32 = 0 \quad \Rightarrow \quad m = \pm \sqrt{32} i \]

\[ x = C_1 \cos \left( \sqrt{32} t \right) + C_2 \sin \left( \sqrt{32} t \right) \quad \Rightarrow \quad x(0) = C_1 = -\frac{1}{2} \]

\[ x' = -\sqrt{32} C_1 \sin \sqrt{32} t + \sqrt{32} C_2 \cos \sqrt{32} t \]

\[ x'(0) = \sqrt{32} C_2 = 2 \quad \Rightarrow \quad C_2 = \frac{2}{\sqrt{32}} \]

\[ x(t) = -\frac{1}{2} \cos \left( \frac{\sqrt{32} t}{\sqrt{32}} \right) + \frac{2}{\sqrt{32}} \sin \left( \frac{\sqrt{32} t}{\sqrt{32}} \right) \]

Amplitude = \[ A = \sqrt{\left( -\frac{1}{2} \right)^2 + \left( \frac{2}{\sqrt{32}} \right)^2} = \sqrt{\frac{1}{4} + \frac{4}{32}} = \sqrt{\frac{8 + 4}{32}} = \sqrt{\frac{12}{32}} = \sqrt{\frac{3}{9}} \]

Period = \[ T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{32}} \]