SHOW ALL YOUR WORK to avoid loss of points.

1. For the following equations indicate their order and indicate all that apply: are separable, linear, exact or neither. (some equations can be more than one kind!) Do NOT solve the equations.

   (No half pts, no need to justify 2 )

   [2pt] a) \( y \frac{d^2 y}{dx^2} + y = 1 + xy \) : 2nd Order; neither \( e.g. \)

   \[ u = y - 6x \quad N = x - 1 \]

   [3pt] b) \( \frac{dy}{dx} = \frac{y - 6x}{1 - x} \) : 1st order; linear and exact

   \[ u = y - 6x \quad N = x - 1 \]

   [2pt] c) \( \frac{dy}{dx} = \frac{e^{x+y}}{x} \) : 1st order; separable

   [3pt] d) \( \frac{dy}{dx} = \frac{3x^2 - 2 \arctan x + e^x}{y^3 + \sec^2 y - \sqrt{y}} \) : 1st order separable and exact

   put \( m = 3x^2 - 2 \arctan x + e^x \quad \text{and} \quad N = y^3 + \sec^2 y + \sqrt{y} \)

   \[ \frac{\partial m}{\partial x} = 0 \quad \text{and} \quad \frac{\partial N}{\partial x} = 0. \]

2. Find the equilibrium solutions of the differential equation \( \frac{dy}{dx} = y(y^2 - 16) \), and classify them as asymptotically stable, unstable or semi-stable.

   Equilibrium Sol. ae \( y = c \) where \( f(c) = 0 \) , with \( f = y(y^2 - 16) \)

   Crit pts: \( c = 0, 4, -4 \)

   Number line \( -4 \quad -1 \quad 1 \quad 4 \)

   Phase line: \( y = 4 \) \( \{ \) unstable

   \( y = -4 \) \( \{ \) and \( y = 0 \) asymptotically stable
3. Find the model function for the population size of a colony of bacteria that is known to triplicate every half an hour and at time zero the population is $P_0$ million of bacteria. (That is, find the general solution $P(t)$ of the corresponding dif.eq. which models this scenario- assume that the rate of growth of the population is proportional to its size at a given time.)

\[ \frac{dp}{dt} = k \cdot P \Rightarrow \frac{1}{P} \cdot dp = k \cdot dt \Rightarrow \ln |P| = kt + c_i \]

\[ \Rightarrow P = C e^{kt} \quad P(0) = C e^{0} = C = P_0 \quad \text{so} \quad P(t) = P_0 e^{kt} \]

Now find $k$:

We know after $t = \frac{1}{2}$, $P$ goes from $P_0$ to $3P_0$, i.e., $3P_0 = P_0 e^{\frac{k}{2}}$

Solve for $k$: $e^{\frac{k}{2}} = 3$ \Rightarrow $\frac{k}{2} = \ln 3$ \Rightarrow $k = 2 \cdot \ln 3$

Ans: \[ P(t) = P_0 e^{(2 \ln 3) \cdot t} \quad \text{or} \quad P(t) = P_0 \cdot 3^{2t} \]

4. Solve the following exact equation and corresponding IVP (no need to verify it is exact):

\[ (y \cos(xy) + e^{x-y}) \, dx + (x \cos(xy) - e^{x-y} + 1) \, dy = 0; \quad y(\sqrt{\pi}) = \sqrt{\pi} \]

\[ M = \int M \, dx = \int y \cos(xy) + e^{x-y} \, dx = \sin(xy) + e^{x-y} + g(y) \]

\[ N = \frac{\partial}{\partial y} \left( \sin(xy) + e^{x-y} + g(y) \right) = x \cos(xy) - e^{x-y} + g'(y) = x \cos(xy) - e^{x-y} + 1 \]

\[ \Rightarrow g'(y) = 1 \quad \Rightarrow g(y) = y \]

Thus \[ f(x,y) = \sin(xy) + e^{x-y} + y \]

Grad f \[ \Rightarrow \sin(xy) + e^{x-y} + y = C \]

IVP: \[ \text{Plug} \quad x = \sqrt{\pi}, \quad y = \sqrt{\pi} \]

\[ \sin \left( \sqrt{\pi} \cdot \sqrt{\pi} \right) + e^{\sqrt{\pi} - \sqrt{\pi} + \sqrt{\pi}} = C \]

\[ \sin^2 \pi + e^0 + \sqrt{\pi} = C \quad \Rightarrow \quad C = 1 + \sqrt{\pi} \]

Sol: \[ \sin(xy) + e^{x-y} + y = 1 + \sqrt{\pi} \]
5. Solve the following equation using an appropriate substitution

\[ \frac{dy}{dx} - \frac{2y}{x} = xy^3 \]

Note this isn't homogeneous nor of the form \( y' = f(Ax + By + c) \), but it is a Bernoulli equation.

\[ \frac{dy}{dx} - \frac{2y}{x} = x \quad \iff \quad \frac{1}{y^3} \left( -\frac{2y}{x} \right) = x \]

let \( u = y^{-2} \) \( \Rightarrow \frac{du}{dx} = -2y^{-3} \frac{dy}{dx} \quad \Rightarrow \quad \int y^{-3} \frac{dy}{dx} = -\frac{1}{2} \int \frac{du}{dx} \]

Substitute: \( -\frac{1}{2} \frac{du}{dx} - \frac{2y}{x} u = x \) \( \iff \) linear in \( u \).

In standard form: \( \frac{du}{dx} + \frac{4}{x} u = -2x \) then \( u = e^{\int \frac{4}{x} dx} e^{\int -2x \frac{dx}{x}} = x^4 \)

The linear equation becomes \( \frac{d}{dx} \left( x^4 u \right) = -2x x^4 \cdot x^4 \cdot u = \int x^5 dx = -\frac{x^6}{6} + C \)

\( u = -\frac{x^2}{3} + CX^{-4} \)

\( y^{-2} = -\frac{x^2}{3} + CX^{-4} \) \( \iff \) Implicit Sol ok

\( y = \left( -\frac{x^2}{3} + CX^{-4} \right)^{-\frac{1}{2}} \) \( \iff \) In fact better as with we lose some information