16.7 Stoke's Theorem

Let S be a surface with boundary C (we write C=∂S). Let \( \mathbf{n} \) be the surface's unit normal vector.

We say C is oriented consistently with \( \mathbf{n} \) if the direction of the parametrization of C and the direction of \( \mathbf{n} \) satisfy the right hand rule (consider that the surface lies on the palm side of your hand).

\[ \oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} \left( \nabla \times \mathbf{F} \right) \cdot \mathbf{n} \, d\sigma \]

**Stoke's Theorem.** Let S be a piecewise smooth oriented surface, let \( \mathbf{n} \) and the boundary of S, C, be oriented consistently, and let \( \mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \) be a vector field with continuous partial derivatives, then:

\[ \oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} \left( \nabla \times \mathbf{F} \right) \cdot \mathbf{n} \, d\sigma \]

Note that if S is a plane surface on the xy-plane, then \( \mathbf{n} = \mathbf{k} \) and \( \left( \nabla \times \mathbf{F} \right) \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \).

\[ \oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA \]

that is, Green's Theorem is a special case of Stoke's Theorem.
## The Big Picture!

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Partial Differentiation/ Chain Rule

Gradient

* Directional Derivatives

* Tangent planes

* Critical points

2nd Derivative Test (Classify Critical Points).

Examples:

1. Decide whether the vector field \( \vec{F} \) below is conservative or not.

\[
\vec{F}(x, y) = (\cos(xy) - xy \sin(xy) - y^2 e^{-x})\hat{i} + (2ye^{-x} - x^2 \sin(xy))\hat{j}.
\]

Is \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \) ?

\[
\frac{\partial M}{\partial y} = -\sin(xy)x - (x\sin(xy) + x^2y \cos(xy)) - 2ye^{-x},
\]

\[
\frac{\partial N}{\partial x} = -2ye^{-x} - (2x\sin(xy) + x^2y \cos(xy)),
\]

Yes, it is conservative.
2. Find \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \), where \( w = xy + yz + xz \) and \( x = u + v, y = u - v \) and \( z = uv \).

\[
\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} \\
= (y+z)(1) + (x+z)(1) + (x+y)v \\
= u-v + uv + u+v + uv + (2u)v \\
= 2u + 2uv + 2uv.
\]

\[
\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v} \\
= (y+z)(1) + (x+z)(-1) + (x+y)u \\
= y-x + (x+y)u = -2v + 2xu^2
\]

3. Find the derivative of \( f(x, y, z) = 3e^x \cos(yz) \) at \( P_0(0, 0, 0) \) in the direction of the vector \( \vec{v} = 2\hat{i} - \hat{j} - 2\hat{k} \).

\[
|\vec{v}| = \sqrt{4 + 1 + 4} = 3
\]

\[
\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}
\]

\[
\vec{v} = \left< 3e^x \cos(yz), -3ze^x \sin(yz), -3ye^x \sin(yz) \right> \\
\vec{v}(0,0,0) = \left< 3, 0, 0 \right>
\]

\[
\hat{D}f(P_0) = \left< \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \right> \cdot \left< 3, 0, 0 \right> = 2
\]

\[
\hat{D}f = \vec{v} \cdot \vec{u} = |\vec{v}| \cos \theta
\]
4. Find the direction in which the function \( f(x, y) = x^2 - xy + y^2 - y \) decreases most rapidly at the point \((1, -1)\).

\[
\text{Ans.} \quad -\vec{\nabla}f(P_0) = -\vec{\nabla}f(1, -1).
\]

\[
\vec{\nabla}f = \langle 2x - y, -x + 2y - 1 \rangle
\]

\[
\vec{\nabla}f(1, -1) = \langle 2(-1), -1 - 2\cdot(-1) \rangle = \langle 3, 4 \rangle
\]

\[
\text{ans} = \langle -3, 4 \rangle
\]

5. Find the tangent plane to the surface \( x^2 + y^2 - z^2 = 18 \) at the point \( P_0(3, 5, -4) \)

\[
F(x, y, z) = 0
\]

\[
\vec{n}_P = \frac{\vec{\nabla}F(P_0)}{|\vec{\nabla}F(P_0)|}
\]

\[
\vec{\nabla}F = \langle 2x, 2y, -2z \rangle_{P_0} = \langle 6, 10, 8 \rangle = 2 \langle 3, 5, 4 \rangle
\]

\[
\vec{n} = \langle A, B, C \rangle_{P_0}
\]

\[
A(x - x_0) + B(y - y_0) + C(z - z_0) = 0
\]

\[
3(x - 3) + 5(y - 5) + 4(z + 4) = 0
\]
6. Find the equation of the tangent plane to the surface \( z = \ln(x^2 + y^2) \) at the point \( P_0(1, 0, 0) \).

\[
F(x, y, z) = \ln(x^2 + y^2) - z = 0
\]

\[
\nabla F|_{P_0} = \left\langle \frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2}, -1 \right\rangle|_{P_0} = \left\langle 2, 0, -1 \right\rangle
\]

\[
2(x-1) - (z-0) = 0
\]

7. Find and classify the critical points of \( f(x, y) = x^3 - y^3 - 2xy + 6 \).

\[
\nabla f = \left\langle 3x^2 - 2y, -3y^2 - 2x \right\rangle
\]

\[
y^2 (3x^2 - 2y = 0) \quad \rightarrow \quad 3x^2 + 2x = 0
\]

\[
x^2 (-3y^2 - 2x = 0) \quad \rightarrow \quad x(3x + 2) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = -\frac{2}{3}
\]

\[
y = -\frac{2}{3}
\]

\[
y^3 = -x^3
\]

\[
\Rightarrow \quad y = -x
\]

\[
f_{xx} = 6x
\]

\[
f_{yy} = -6y
\]

\[
f_{xy} = -2
\]

\[
D(x, y) = f_{xx} f_{yy} - (f_{xy})^2
\]

\[
D(0, 0) = -4 \quad \text{(Saddle pt at (0, 0))}
\]

\[
D(\pm\frac{2}{3}, \pm\frac{2}{3}) = -12 > 0 \quad \text{local max}
\]

\[
f_{xy}(\pm\frac{2}{3}, \pm\frac{2}{3}) < 0
\]

\[
\text{at } (\pm\frac{2}{3}, \pm\frac{2}{3})
\]