Ch. 2 Logic

The study of logic starts with "statements" i.e. sentences or mathematical expressions that are either definitely true or definitely false. We often denote statements with:

\[ \text{P, Q, R, S...} \]

**Ex.**

\[ \text{P: Every even number is divisible by 2,} \]

\[ \text{Q: } 2 \in \mathbb{Z} \]

\[ \text{R: } \sqrt{3} \notin \mathbb{Z} \]

\[ \text{S: } \mathbb{N} \subseteq \mathbb{Z} \]

\[ \text{S: Some right triangles are isosceles} \]

(\text{False Statements})

\[ \text{p: All triangles are isosceles} \]

\[ \text{p: } 5 = 7 \]

\[ \text{p: } \mathbb{Z} \subseteq \mathbb{N} \]

**Sentences that are not statements:**

\[ \text{Z, 42, Add 5 to both sides, what is the solution of } 3x = 1 \]

**Statements can contain variables (open statements)**

e.g., \( P(x): x \) is an even number

**Ex. Fermat's Last Theorem:**

\[ \text{For all numbers } a, b, c, n \in \mathbb{N} \text{ for } n > 2 \text{ it is the case that } a^n + b^n \neq c^n \]

**Goldbach Conjecture**

\[ \text{Every even integer greater than 2 is a sum of two prime numbers} \]
2.2. **And, or, not**

**And & or:** Used to combine statements to form new statements.

\[ R: \text{The number } 2 \text{ is even AND the number } 3 \text{ is odd.} \]

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<th>( P \land Q )</th>
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\( \land \) Denotes **AND**

\( \lor \) Denotes **OR**

Note "or" does not mean one or the other; i.e., both can be true.

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\( \neg \) Denotes **NOT**; If a statement is not true, use \( \neg \)

*E.g.*, \( P \) is not true \( \neg P \)

\( P_1: 2 \) is even \( \neg P_2: 4 \) is prime

\( \neg P_1: 2 \) is not even \( \neg P_2: 4 \) is not prime

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2.3 Conditional Statements

E.g. P: the integer \( a \) is a multiple of 6
Q: the integer \( a \) is divisible by 2

R1: If the integer \( a \) is a multiple of 6 then
it is divisible by 2.
R2: If \( P \), then \( Q \)
P \( \Rightarrow \) Q
P implies Q

Truth Table

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"P \( \Rightarrow \) Q" P is a sufficient condition for Q (but not necessary)
Q is a necessary condition for P

Ex.
For a function to be continuous it is sufficient that it is differentiable
P: \( f \) is differentiable
Q: \( f \) is continuous

2.4 Biconditional Statement

If \( P \Rightarrow Q \), then \( Q \Rightarrow P \) is not necessarily true

Def. The conditional statement \( Q \Rightarrow P \) is called the converse

If both the conditional and its converse are true
it is called a Biconditional

E.g. \( P \iff Q \): \( (P \Rightarrow Q) \land (Q \Rightarrow P) \)
"P if and only if Q"