Strong Induction

Prove that an n-cent postage (n ≥ 12) can be made up using only 3-cent and 7-cent stamps.

Observe that if we assume the statement is true for n, it is hard to show that it's true for n+1. E.g., 20-cent postage: 7 + 3 + 7 + 3 but 21-cent: 7 + 7 + 7.

We will instead do an induction step of 3 units, and we need 3 base cases.

Base case 1a: 12 = 3 + 3 + 3 + 3 \( \checkmark \)
Base case 1b: 13 = 3 + 3 + 7 \( \checkmark \)
Base case 1c: 14 = 7 + 7 \( \checkmark \)

Case 15: base case 12 plus 3.
Case 16: base case 13 plus 3.
And so on...

Assume \( n = 3k + 7 \) for \( k \in \mathbb{N} \).
\[ n + 3 = (3k + 7) + 3 \]
\[ = 3(k + 1) + 7 \]

Proofs by Smallest Counterexample

Combination of induction & contradiction.

Prop. \( S_k \), \( k \in \mathbb{N} \) are true.

1) Show \( S_1 \) is true.
2) Suppose, for contradiction, \( S_k \) not true for every \( n \).
3) Let \( k > 1 \) be the smallest integer such that \( S_k \) is false.
4) \( S_{k-1} \) is true, but \( S_k \) is false (use this to get a contradiction).
Ex. If \( n \in \mathbb{N} \), then \( 4 | (5^n - 1) \).

1. \( S_1: 4 | (5^1 - 1) \implies 4 | 4 \) is clearly true.
2. Assume not for every \( n \in \mathbb{N} \).
3. Let \( k > 1 \) be the smallest integer for which \( 4 | (5^k - 1) \).
4. We know \( 4 | (5^{k-1} - 1) \), so we can write:
   \[ 5^{k-1} - 1 = 4a \quad \text{for some } a \in \mathbb{Z} \]

Multiply by \( S \): \( 5^k - 5 = 20a \)
\[ 5^k - 1 = 20a + 4 \]
\[ = 4(5a + 1) \]

This contradicts \( \times \).

Fundamental Theorem of Arithmetic

Any integer greater than 1 has a unique prime factorization. That is: \( \forall n \geq 1, n \in \mathbb{N}, n = p_1 \cdots p_k \) where each \( p_i \) is a prime number and if \( n = a_1 \cdots a_k \) where each \( a_i \) is a prime number then \( k = \) and:
\[ \{ p_i : i = 1, \ldots, k \} = \{ a_i : i = 1, \ldots, k \} \]

Proof: We'll show first that \( n \) has a prime factorization by strong induction.

Base step: \( n = 2 \) is already factored into primes.

Inductive step: Assume every integer \( \leq n \) has a prime factorization.

Now consider \( n + 1 \), if \( n + 1 \) is a prime we're done.
Otherwise, \( n + 1 = ab \), where \( a, b < n \). By induction hypothesis, we can write \( a = p_1 \cdots p_k \) and \( b = q_1 \cdots q_l \) where \( p_i, q_j \) are prime numbers.

\[ n + 1 = ab = p_1 \cdots p_k q_1 \cdots q_l \] is a prime factorization for \( n + 1 \).

Now we prove uniqueness, by smallest counterexample.
Suppose $n > 2$ is the smallest integer for which the factorization is not unique.

$$n = p_1 \cdots p_k \quad \text{and} \quad n = q_1 \cdots q_l$$

From (1) we see that $p_1 | n$, so that by (2), $p_1 | a_1 \cdots a_i$.

Then by previous proposition, $p_1 | a_i$ for some $a_i, i \in \mathbb{N}^+$.

$a_i$ is prime, so $p_1 = a_i$.

$$p_1 \cdots p_k = a_1 \cdots a_l$$

Divide by $p_1$: $p_2 \cdots p_k = a_1 \cdots a_i, a_i, \cdots a_l$

But $m < n$ and it also has 2 distinct prime factorizations, contradicting our assumption.

Thus the prime factorization for every $n > 2, n \in \mathbb{N}$ is unique.

More on Fibonacci Numbers

Recall $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$.

Prove The Fibonacci sequence satisfies:

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n \star$$

Base Case: $n = 1$, $F_0 = F_{n+1} = 1$.

$$F_2^2 - F_2F_1 - F_1^2 = (-1)^1$$

$$-1 = -1 \checkmark$$
Assume \( A \) is true and show \( F_{n+2}^2 - F_{n+1}F_{n+4} - F_n^2 = (-1)^{n+1} \).

\[
F_{n+2}^2 - F_{n+1}F_{n+4} - F_n^2 = (F_n + F_{n+1})^2 - (F_n^2 + F_{n+1}F_{n+2}) - F_n^2
\]

\[
= F_n^2 + 2F_nF_{n+1} + F_{n+1}^2 - F_n^2 - F_{n+1}F_{n+2} - F_n^2
\]

\[
= F_n^2 + 2F_nF_{n+1} - F_{n+1}^2
\]

\[
= - (F_{n+1}^2 - F_n^2)
\]

\[
= - (-1)^n
\]

\[
= (-1)^{n+1} \square
\]

Divide \( A \) by \( F_n^2 \):

\[
\frac{(F_{n+1})^2 - F_{n+1}F_n - 1}{F_n^2} = \frac{(-1)^n}{F_n^2}
\]

Take the limit:

\[
\lim_{n \to \infty} \left[ \frac{(F_{n+1})^2 - F_{n+1}F_n - 1}{F_n^2} \right] = \lim_{n \to \infty} \left[ \frac{(-1)^n}{F_n^2} \right]
\]

Let \( L = \lim_{n \to \infty} \frac{F_{n+1}}{F_n} \).

We have \( L^2 - L - 1 = 0 \).

\[
L = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} = \Phi \quad \text{(Golden Ratio)}. \square
\]