Math 201 Notes 2/23

Chapter 10 Mathematical Induction

Outline for a proof by induction:
Prop. The statements $s_1, s_2, s_3, \ldots$ are all true.
(Sn is true for all $n \in \mathbb{N}$)

1st Base Case: i.e. prove $s_1$ is true.

2nd Inductive Step: prove $s_k \Rightarrow s_{k+1}$

Assume $s_k$ which is called the inductive hypothesis

Prop. If $n \in \mathbb{N}$ then $\sum_{k=1}^{n} (2k-1) = n^2$

Proof

(Base Step) 1) $s_1 = \sum_{k=1}^{1} (2k-1) = 2(1) - 1 = 1 = 1^2 = n^2$

(Inductive Step) 2) Assume $s_n$ is true, that is

$s_n = \sum_{k=1}^{n} (2k-1) = n^2$

$s_{n+1} = \sum_{k=1}^{n+1} (2k-1) = \sum_{k=1}^{n} (2k-1) + (2(n+1)-1)$

$= n^2 + 2n - 1 \quad \text{(by inductive hypothesis)}$

If follows by induction that $\sum_{k=1}^{n} (2k-1) = n^2$

Prop. If $n$ is non-negative integer then $5 \mid n^5 - n$

Proof

1) Base Step corresponds to $n = 0$, does $5 \mid 0^5 - 0$, yes

2) Inductive Step. Assume $5 \mid n^5 - n$, and show $5 \mid (n+1)^5 - (n+1)$

$(n+1)^5 - (n+1) = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - n - 1$

$= n^5 - n + 5(n^4 + 2n^3 + 2n^2 + n)$

(Ind hyp) $5 \mid 5x + 5y = 5(x+y) \in Z \quad \text{where} \quad x+y \in Z$

$5 \mid (n+1)^5 - (n+1)$
Prop. For each $n \in \mathbb{N}$, it follows that $2^n \leq 2^{n+1} - 2^{n-1} - 1$

Proof

(Base) 1) Is it true for $n=1$? $2 \leq 2 - 1 - 1 = 2$
   Yes, $2 \leq 2$ is true.

2) Assume for $k \geq 1$, $S_k$ is true.
   $2^k \leq 2^{k+1} - 2^{k-1} - 1$ holds, now show it's true for $k+1$.
   
   
   

Prop. Suppose $a_1, a_2, \ldots, a_n$ are $n$ integers, $n \geq 2$.

If $p$ is prime and $p | a_1 a_2 \ldots a_n$, then $p | a_i$ for at least one $i \in \{1, 2, \ldots, n\}$

Proof

(Base) 1) $n=2$ Proof. $p | a_1 a_2$, then $p | a_1$ or $p | a_2$

   $p | a_1 a_2 \Rightarrow a_1 a_2 = px, x \in \mathbb{Z}$

   Suppose $p \nmid a_1$, then $gcd(p, a_1) = 1$.

   We can write $gcd(p, a_1) = pk + a_1 l = 1$ for some $k, l \in \mathbb{Z}$.

   Multiplying by $a_2$, $pa_2 k + a_1 a_2 l = a_2$

   $pa_2 k + pxl = a_2 \Rightarrow p | a_2$

   $pl(a_2 k + xl) = a_2 \Rightarrow p | a_2$

2) Assume $p | a_1 \ldots a_k \Rightarrow p | a_i$ for at least one $a_i$.

   Prove. $[p | a_1 \ldots a_k \Rightarrow p | a_i \text{ for at least one } a_i]$

   Suppose $p | (a_1 a_2 \ldots a_k) a_{k+1}$ from base case we know either $p | a_1 \ldots a_k$ or $p | a_{k+1}$.

   If $p | a_{k+1}$ (we're done), otherwise by induction hypothesis then $p | a_i$ for some $i \in \{1, 2, \ldots, k\}$
Recursive Formulas

Geometric Sum: \[ S_n = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r} \]

1) \[ S_1 = \sum_{k=0}^{1} r^k = r^0 + r^1 = 1 + r \]
\[ \frac{1-r^2}{1-r} = \frac{(1-r)(1+r)}{1-r} = 1 + r \] Same

2) Assume \[ S_n = \frac{1-r^{n+1}}{1-r} \] holds, show \[ S_{n+1} \] is true.

\[ S_{n+1} = S_n + r^{n+1} \]
\[ = \frac{1-r^{n+1}}{1-r} + r^{n+1} \] (by ind. hyp)
\[ = \frac{1-r^{n+1} + r^{n+1} - r^{n+2}}{1-r} \]
\[ = \frac{1-r^{n+2}}{1-r} \]

Strong Induction

Prop. \[ S_n \] is true for all \( n \in \mathbb{N} \)
1) Prove \( S_1 \) (or first several \( S_n \))
2) Ind. Step \( (S_1 \land S_2 \land \ldots \land S_k) \Rightarrow S_{k+1} \)

Prop. Suppose \( a_1 = 1, a_2 = 8 \), for \( n \geq 3 \) \( a_n = a_{n-1} + 2a_{n-2} \)
Prove \( a_n = 3 \cdot 2^n - 2(-1)^n \)
Base: \( n = 3 \) \( a_3 = a_2 + 2a_1 = 8 + 2 = 10 \) on the other hand \( a_3 = 3 \cdot 2^2 - 2(-1) = 12 - 2 = 10 \)