(16.2) From last time...

\[ \int \vec{F} \cdot \, ds = \int \vec{F} \cdot d\vec{r} = \int M \, dx + N \, dy + P \, dz \]

where \( \vec{F} = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle \) and the curve \( C \) is \( \vec{r}(t) = \langle x(t), y(t), z(t) \rangle, a \leq t \leq b \).

When \( \vec{F} \) is a force, we interpret the integral as work. When \( \vec{F} \) is the velocity of a liquid, the integral is called the flow along \( C \).

If the liquid goes across \( C \) we get the Flux:

\[ \text{Flux} = \int \vec{F} \cdot \vec{n} \, ds \quad \text{(for plane curves i.e., 2-variables)} \]

We saw that

\[ \vec{n} = \frac{\vec{r}}{r} \times \hat{k} = \left( \frac{d\vec{r}}{ds} \right) \times \hat{k} = \left( \frac{dx}{ds} \hat{i} + \frac{dy}{ds} \hat{j} \right) \times \hat{k} \]

\[ = \frac{dx}{ds} (\hat{i} \times \hat{k}) + \frac{dy}{ds} (\hat{j} \times \hat{k}) \]

\[ \vec{n} = \frac{dy}{ds} \hat{i} - \frac{dx}{ds} \hat{j} \]
Thus \( \mathbf{F} \cdot \mathbf{n} = \langle M(x,y,z), N(x,y,z) \rangle \cdot \langle \frac{dy}{ds}, -\frac{dx}{ds} \rangle \)

\[ = M \frac{dy}{ds} - N \frac{dx}{ds} \]

\[ \Rightarrow \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C M \, dy - N \, dx = -\oint_C \]

- \( C \) is \( C \) parameterized in opposite dir.

Recall \( C \) is a closed curve (boundary of some region)

Thus the notation \( \oint_C \)

E.g. Find the flux of \( \mathbf{F} = (x-y) \mathbf{i} + x \mathbf{j} \) across the circle \( x^2 + y^2 = 1 \) in the xy-plane

\[ \mathbf{r}(t) = \langle \cos t, \sin t \rangle \quad M = x-y \quad N = x, \quad 0 \leq t \leq 2\pi \]

\[ \text{Flux} = \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_C (x-y) \, dy - (x) \, dx \]

\[ dy = \cos t \, dt \]
\[ dx = -\sin t \, dt \]

\[ = \int_0^{2\pi} \left[ (\cos t - \sin t) \cos t - \cos t (-\sin t) \right] \, dt \]

\[ = \int_0^{2\pi} \cos^2 t \, dt = \int_0^{2\pi} \frac{1}{2} + \frac{\cos 2t}{2} \, dt \]

\[ = \left[ \frac{1}{2} t + \frac{\sin 2t}{4} \right]_0^{2\pi} = \frac{2\pi}{2} = \pi \]
16.3 Path Independence & Conservative Fields.

In single variable calculus the main tool to evaluate integrals is the Fundamental Theorem of Calculus.

\[ \int_{a}^{b} f'(x) \, dx = f(b) - f(a) \]

Is there a similar result for line integrals?

YES!

We'll use \( \nabla f \) instead of \( f' \).

Suppose \( f = f(x(t), y(t)) \), then via chain rule

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right> \cdot \left< \frac{dx}{dt}, \frac{dy}{dt} \right> = \nabla f \cdot \vec{v}'(t)
\]

\[
\int_{C} \nabla f \cdot d\vec{r} = \int_{a}^{b} \nabla f \cdot \vec{r}'(t) \, dt = \int_{a}^{b} \frac{df(\vec{r}(t))}{dt} \, dt = f(\vec{r}(b)) - f(\vec{r}(a))
\]
Fundamental Theorem of Line Integrals. Let \( C \) be a smooth curve joining the point \( A \) to the point \( B \) in the plane or in space and parametrized by \( \vec{r}(t) \). Let \( f \) be a differentiable function with continuous gradient \( \vec{F} = \nabla f \) on a domain \( D \) containing \( C \). Then

\[
\int_{C} \vec{F} \cdot d\vec{r} = f(B) - f(A)
\]

**Definitions**

Let \( \vec{F} \) be a vector field, if \( \int_{C} \vec{F} \cdot d\vec{r} \) is the same for all possible paths from \( A \) to \( B \), we say \( \vec{F} \) is **conservative** (i.e. \( \int_{C} \vec{F} \cdot d\vec{r} \) is path independent).

In fact (Theorem): \( \vec{F} \) is conservative if and only if \( \vec{F} = \nabla f \) for some differentiable function \( f \).

We say \( f \) is a potential function for \( \vec{F} \).
Example

Note that the vector field \( \vec{F} = \langle yz, xz, xy \rangle \) is a gradient, indeed a potential function for \( \vec{F} \) is \( f = xyz \) \( \nabla f = \vec{F} \).

We can then easily find the work done by \( \vec{F} \) from the point \( A(-1,3,9) \) to \( B(1,6,-4) \).

\[
W = \int_{C} \vec{F} \cdot d\vec{r} = \int_{C} \nabla f \cdot d\vec{r} = f(B) - f(A)
\]

\[
= (1)(6)(-4) - (-1)(3)(9)
\]

\[
= -24 - (-27) = 27 - 24 = 3
\]

Note that when \( C \) is a closed curve, \( \oint_{C} \nabla f \cdot d\vec{r} = 0 \).

Theorem

The following statements are equivalent.

1) \( \vec{F} = \nabla f \) for some \( f \)

2) \( \int_{C} \vec{F} \cdot d\vec{r} \) is path independent

3) \( \int_{C} \vec{F} \cdot d\vec{r} = 0 \) for all closed curves \( C \).
When shall we use the F.Thm for line integrals? i.e., how do we know if the vector field $\vec{F} = \nabla f$ for some $f$.

**Component Test for Conservative Fields**

Let $\vec{F} = \langle M, N, P \rangle$ be a vector field, * partial derivatives exist, Domain open & simply connected.

Then $\vec{F}$ is conservative (i.e. a gradient) if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

When $\vec{F} = \langle M, N, P \rangle$ is conservative we say that the differential form $Mdx + Ndy + Pdz$ is exact.

Once we know $\vec{F}$ is conservative we need to find a potential function $f$. (i.e. $f$ so that $\vec{F} = \nabla f$)
Example The vector field \( \vec{F} = \langle e^x \cos y + yz, xz - e^x \sin y, xy + z \rangle \) is conservative (check at home!), find a potential function for it.

Find \( f(x,y,z) \) so that

\[
\frac{\partial f}{\partial x} = e^x \cos y + yz, \quad \frac{\partial f}{\partial y} = xz - e^x \sin y, \quad \frac{\partial f}{\partial z} = xy + z
\]

\( \Rightarrow f = \int (e^x \cos y + yz) \, dx = e^x \cos y + xyz + g(y,z) \)

\( \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (e^x \cos y + xyz + g(y,z)) = xz - e^x \sin y \)

\[\begin{align*}
-e^x \sin y + xz + \frac{\partial}{\partial y} g(y,z) &= xz - e^x \sin y \\
\frac{\partial}{\partial y} g(y,z) &= 0 \quad \Rightarrow \quad g = \hat{g}(z)
\end{align*}\]

\( \Rightarrow \frac{\partial f}{\partial z} = \frac{\partial}{\partial z} (e^x \cos y + xyz + \hat{g}(z)) = xy + z \)

\[\begin{align*}
xy + \hat{g}'(z) &= xy + z \\
\hat{g}'(z) &= z \quad \Rightarrow \quad \hat{g}(z) = \frac{z^2}{2} + C
\end{align*}\]

\( \Rightarrow f(x,y,z) = e^x \cos y + xyz + \frac{z^2}{2} + C \)

We need \( \int_C \vec{F} \cdot d\vec{r} = f(B) - f(A) = f(1,1,1) - f(1,0,1) \)

\[\begin{align*}
&= (e \cos 1 + 1 + \frac{1}{2} + C) - (e + 0 + \frac{1}{2} + C) \\
&= e \cos 1 - e + 1
\end{align*}\]
Evaluate \( \int_C y \, dx + x \, dy + 4 \, dz \), where \( C = C_1 \cup C_2 \cup C_3 \).

- \( C_1 \): The line segment from \((1, 1, 1)\) to \((1, 1, 0)\).
- \( C_2 \): The line segment from \((1, 1, 0)\) to \((2, 3, 0)\).
- \( C_3 \): The line segment from \((2, 3, 0)\) to \((2, 3, -1)\).

It's worth trying to find \( f \) so that

\[
\langle M, N, P \rangle = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)
\]

\( f = \int y \, dx = yx + g(y, z) \)

\[
\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x \quad \Rightarrow \quad \frac{\partial g}{\partial y} = 0 \quad \Rightarrow \quad g = g(z)
\]

\[
\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(yx + g(z)) = g'(z) = 4 \quad \Rightarrow \quad g(z) = 4z
\]

\( f = xy + 4z \).

\[
\int_C \ldots = f(2, 3, -1) - f(1, 1, 1)
\]

\[
= \left[(2)(3) - 4\right] - \left[(1)(4) + 4\right]
\]

\[
= 6 - 8 = -2
\]

\[
= 2 - 5 = -3
\]