16.1 Line Integrals

Recall in 2D, integrals are taken over an interval (a line segment)

\[ \int_a^b f(x) \, dx \]

Now we'd like to integrate along a general curve segment, say with parametrization \( \vec{r}(t) ; a \leq t \leq b \), for instance.

An application of such integral would be finding mass, and center of mass of a wire shaped in the form of \( \vec{r}(t) \).

We could also interpret the integral of \( f \) along a flat curve, as the area of the surface spanned by the curve.

When \( f=1 \), such area would be the length of \( \vec{r}(t) \), this is the "arc length" integral.
In the area case we can approximate with a Riemann Sum:

Take a partition of the curve, that corresponds to a partition of \([a,b]\) for the parameter \(t\).

\[
A \approx \sum_{k=1}^{n} f(\vec{r}(t_k)) \Delta S_k
\]

We could also approximate the mass of a wire in the shape of \(\vec{r}(t)\)

\[
M \approx \sum_{k=1}^{n} \delta(\vec{r}(t_k)) \Delta S_k
\]

In the limit (as the partition gets finer) we get what we call line integrals.

**Definition**    If \(f\) is defined on a curve \(C\) given parametrically by:

\[
\vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \leq t \leq b
\]

then the line integral of \(f\) over \(C\) is:

\[
\int_{C} f(x, y, z) \, ds = \lim_{n \to \infty} \sum_{k=1}^{n} f(\vec{r}_k) \Delta S_k
\]
Recall: Speed = \[ |v| = \frac{ds}{dt} \quad \Rightarrow \quad ds = |v| \, dt \]

Thus changing variable from \( ds \) to \( dt \) we get:

\[
\int_C f(x,y,z) \, ds = \int_a^b f(x(t), y(t), z(t)) \, |v(t)| \, dt = \int_a^b f(\vec{r}(t)) \, |\vec{r}'(t)| \, dt
\]

Note that our formula for arc length is the case \( f = 1 \)

When the curve is described, but no parametric equations are given, we need to find a “smooth” parametrization \( \vec{r}(t) \).

Example. Evaluate the integral \( \int_C (y^2 + xz) \, ds \), where \( C \) is the straight line segment from \((0,0,0)\) to \((1,1,1)\).

Parametrization: \(
\vec{r}(t) = (1-t) \vec{r}_0 + t \vec{r}_f, \quad t \in [0,1]
\)

\[
\vec{r}(t) = \begin{cases} 
(1-t)\mathbf{0} + t \mathbf{1} & , t \in [0,1] \\
(1-t)\mathbf{v} + t \mathbf{v} & 
\end{cases}
\]

\[
\vec{r}'(t) = \begin{cases} 
\mathbf{0} & , t \in [0,1] \\
\mathbf{v} & 
\end{cases}
\]

\[
|\vec{r}'(t)| = \sqrt{1+1+t^2} = \sqrt{3}
\]

\[
\int_C (y^2 + xz) \, ds = \int_0^1 (t^2 + t \cdot t) \sqrt{3} \, dt = 2\sqrt{3} \int_0^1 t^2 \, dt
\]

\[
= 2\sqrt{3} \left[ \frac{t^3}{3} \right]_0^1 = \frac{2\sqrt{3}}{3}
\]
b) If we now let $C$ be the union of the line segments:

$C_1$: segment from $(0,0,0)$ to $(1,1,0)$

$C_2$: segment from $(1,1,0)$ to $(1,1,1)$

We need a parametrization for each of the line segments:

$C_1$: $\vec{r}_1 = (1-t)\langle 0,0,0 \rangle + t\langle 1,1,0 \rangle = \langle t,t,0 \rangle$; $0 \leq t \leq 1$

$C_2$: $\vec{r}_2 = (1-t)\langle 1,1,0 \rangle + t\langle 1,1,1 \rangle = \langle 1,1,t \rangle$; $0 \leq t \leq 1$

$$\int_C y^2 + xz \, ds = \int_{C_1} y^2 + xz \, ds + \int_{C_2} y^2 + xz \, ds$$

$|\vec{r}_1'| = \sqrt{2}$

$|\vec{r}_2'| = 1$

$$= \int_0^1 t^2 \sqrt{2} \, dt + \int_0^1 t + t \, dt$$

$$= \frac{\sqrt{2}}{3} t^3 \bigg|_0^1 + (t + \frac{t^2}{2}) \bigg|_0^1 = \frac{\sqrt{2}}{3} + 1.5$$
As mentioned earlier, we could find moments, mass and center of mass of a wire in the shape of a curve $C$.

Thus:

Mass: \[ M = \int_C s \, ds \]

Moments:
\[ M_{yz} = \int_C x \, s \, ds \]
\[ M_{xz} = \int_C y \, s \, ds \]
\[ M_{xy} = \int_C z \, s \, ds \]

Center of Mass:
\[ \bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M} \]
Example. A thin wire is bent in the shape of a semicircle with radius 7. (centered at the origin, use upper semicircle) Its density is proportional to the distance to the x-axis, find the center of mass of the wire.

Parametrization \( \mathbf{r}(t) = \langle 7 \cos t, 7 \sin t \rangle \)

\[ 0 \leq t \leq \pi \]

Density: \( \delta = ky \)

\[
\mathbf{y} = \frac{M_x}{M} = \frac{\int_C y \, ds}{\int_C s \, ds}
\]

\[
|\mathbf{r}'| = \sqrt{49(\sin^2 t + \cos^2 t)} = 7
\]

\[
M = \int_C ky \, ds = k\int_0^\pi 7 \sin t \, 7 \, dt = 49k \left[ -\cos t \right]_0^\pi = 2(49)k = 98k
\]

\[
M_x = \int_C (ky) y \, ds = k\int_0^\pi 49 \sin^2 t \, 7 \, dt = k \frac{7^3}{2} \left[ \frac{1}{2} t - \frac{\sin 2t}{4} \right]_0^\pi = \frac{7^3 \pi}{2} \cdot \frac{k}{2} = \frac{7^3 \pi}{4} \cdot \frac{k}{2}
\]

\[
\mathbf{y} = \frac{\frac{7^3 \pi}{4} \cdot \frac{1}{2}}{2(49)k} = \frac{7 \pi}{4} \cdot \frac{y}{x} = 0, \frac{7 \pi}{4}
\]

Note: \( \bar{x} = 0 \)

By symmetry.