Now the region of integration will be a solid (3-D region).

Consider integrating over a solid the function \( f(x) = 1 \)
(and interpret the integral as a volume).

To define the triple integral over a box:

\[
B = \{(x,y,z) : a_i \leq x \leq a_2 ; b_1 \leq y \leq b_2 ; c_1 \leq z \leq c_2 \}
\]

We take a partition of \( B \) into smaller boxes

\[
\Delta V_k = (\Delta x_k \Delta y_k \Delta z_k) = 1
\]

Then

\[
V \approx \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} (\Delta x_k \Delta y_k \Delta z_k) \rightarrow \lim_{n \to \infty} \int_{c_1}^{c_2} \int_{b_1}^{b_2} \int_{a_i}^{a_2} 1 \, dx \, dy \, dz
\]

Similarly integrating any function \( f(x,y,z) \) over \( B \) we get:

\[
\iiint_B f \, dV
\]
We'll have two main kinds of regions.

Boxes (as defined above), where all the variables are bounded by constants.

Example \( B = \{(x, y, z) : 1 \leq x \leq 2, 0 \leq y \leq 1, -1 \leq z \leq 2\} \)

\[
\iiint_B f \, dV = \int_1^2 \int_0^2 \int_{-1}^1 f \, dx \, dy \, dz
\]

More general solids, where one or two of the variables can be bounded by functions. Depending on the order of integration the different forms can be summarized with:

\[
\int_a^b \left( \int_{g_1(\cdot)}^{g_2(\cdot)} \int_{f_1(\cdot, \cdot)}^{f_2(\cdot, \cdot)} f(x, y, z) \, d\gamma \right) \, dV
\]

Note that there are 6 options for \( dV \)

<table>
<thead>
<tr>
<th>( dV )</th>
<th>( g_i(\cdot) )</th>
<th>( f_i(\cdot, \cdot, \cdot) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( dz , dy , dx )</td>
<td>( g_i(x) )</td>
<td>( f_i(x, y) )</td>
</tr>
<tr>
<td>( dz , dx , dy )</td>
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<td>( f_i(y, z) )</td>
</tr>
</tbody>
</table>
Example. Identify the region given the integral:

\[
\int_0^5 \int_0^{3x} \int_0^{x+2} 4 \, dz \, dy \, dx
\]

(sketch the region)

Example.

Find limits for a triple integral over the region in the 1st octant that is bounded by the parabolic cylinder \( z = 2 - \frac{1}{2} x^2 \) and the plane \( y = x \).

Let's do:

\[
dz \, dy \, dx
\]

\[
0 \leq x \leq 2
\]

\[
0 \leq y \leq x
\]

\[
0 \leq z \leq 2 - \frac{1}{2} x^2
\]
How do we approach finding the limits of integration?

If the order of integration is \( dV = dz \, dy \, dx \) or \( dV = dz \, dx \, dy \), project the solid onto xy-plane. The limits on \( z \) are the upper and lower bounds of the solid.

Upper limit for \( z \) (cone)

Lower limit for \( z \) (parallelepiped).

Region on xy-plane give limits for \( x, y \).

Similarly if \( dV = dy \, dx \, dz \) or \( dV = dy \, dz \, dx \), project onto xz-plane.

For the last two combinations, we project onto yz-plane, this is when

\[
dV = dx \, dy \, dz \quad \text{or} \quad dV = dx \, dz \, dy
\]

This will be better explained through the examples...
Example

For the same region in the previous example, find limits to integrate in the order \( dV = dy \, dx \, dz \)

\[
0 \leq z \leq 2 \quad ; \quad 0 \leq x \leq \sqrt{4-2z} \quad ; \quad 0 \leq y \leq x
\]

\[
\int_{0}^{2} \int_{0}^{\sqrt{4-2z}} \int_{y}^{x} f \, dy \, dx \, dz.
\]

Next do the order \( dV = dx \, dz \, dy \)

\[
z = 2 - \frac{1}{2} x^2 \quad \text{and} \quad y = x \quad ; \quad \text{combine} \quad z = 2 - \frac{1}{2} y^2
\]

\[
2 \quad \rightarrow \quad 2 - \frac{1}{2} y^2 = z \quad ; \quad 0 \leq y \leq 2 \quad ; \quad 0 \leq z \leq 2 - \frac{1}{2} y^2
\]

Bounds for \( x \): \( y \leq x \leq \sqrt{4-2z} \)

\[
\int_{0}^{2} \int_{0}^{2 - \frac{1}{2} y^2} \int_{y}^{\sqrt{4-2z}} f \, dx \, dx \, dy.
\]
Example.
a) Sketch the region $D$ and write a triple integral to find its volume.

\[ D = \{ (x, y, z) : 0 \leq x \leq 1, \ 0 \leq y \leq 3, \ 0 \leq z \leq \frac{1}{6} (12 - 3x - 2y) \} \]

b) Switch the order of integration to $dV = dy \ dz \ dx$