If the surface is of the form \( z = f(x, y) \) we can rewrite:

\[
F(x, y, z) = f(x, y) - z = 0
\]

and use the previous definition for the equation of the tangent plane:

Note \( \frac{\partial F}{\partial x} = f_x, \frac{\partial F}{\partial y} = f_y, \frac{\partial F}{\partial z} = -1 \)

where

\[ Z_0 = f(x_0, y_0) \]

\[
f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) - (z-z_0) = 0
\]

**Example.** Find the plane tangent to the surface \( z = x \cos y - y e^x \) at \((0,0)\)

\[
f_x|_{(0,0)} = \cos y - y e^x = \cos 0 = 1
\]

\[
f_y|_{(0,0)} = -x \sin y - e^x = -e^0 = -1
\]

\[
1 \cdot (x-0) - 1 \cdot (y-0) - (z-0) = 0
\]

\[
\boxed{x - y - z = 0}
\]

**Example.** The surfaces \( f(x, y, z) = x^2 + y^2 - 2 = 0 \) and \( g(x, y, z) = x + z - 4 = 0 \) meet in an ellipse \( E \). Find parametric equations for the line tangent to \( E \) at \( P_0(1,1,3) \).

\( E \) lies on \( f(x, y, z) = 0 \) \( \Rightarrow \) tangent line to \( E \) at \( P_0 \) is \( \perp \) to \( \nabla f|_{P_0} \)

\( E \) lies on \( g(x, y, z) = 0 \) \( \Rightarrow \) tangent line to \( E \) at \( P_0 \) is \( \perp \) to \( \nabla g|_{P_0} \)

Thus the direction of the tangent line is \( \parallel \) to \( \nabla f \times \nabla g \) at \( P_0 \).

\[
\nabla f|_{P_0} = \langle 2x, 2y, 0 \rangle |_{P_0} = \langle 2, 2, 0 \rangle
\]

\[
\nabla g|_{P_0} = \langle 1, 0, 1 \rangle
\]
Estimating change in $f$ in a direction $\mathbf{u}$

Recall in single variable the meaning of a differential:

\[ \Delta y = \Delta f \]
\[ ds = \Delta s \]
\[ \frac{df}{ds} = m \implies df = f'(s) \, ds \]
\[ \Delta f \approx df \]

Similarly, the directional derivative will help us estimate small changes in multivariable functions:

\[ df = (D_{\mathbf{u}} f) \, ds = (\nabla f \cdot \mathbf{u}) \, ds \]

**Example.** Estimate how much the value of $f(x,y,z) = y \sin x + 2yz$ will change if the point $P(x,y,z)$ moves 0.1 unit from $P_1(0,1,0)$ straight towards $P_2(2,2,-2)$

\[
\vec{P_1P_2} = \langle 2, 1, -2 \rangle, \quad \mathbf{u} = \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle
\]

\[
|\vec{P_1P_2}| = \sqrt{4 + 1 + 4} = 3
\]

\[
\nabla f = \langle y \cos x, \sin x + 2z, 2y \rangle
\]

\[
\nabla f|_{P_1} = \langle 1, 0, 2 \rangle
\]

\[
\Rightarrow D_{\mathbf{u}} f = \langle 1, 0, 2 \rangle \cdot \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle
\]

\[
= \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}
\]

\[
\Delta f \approx df = D_{\mathbf{u}} f \cdot ds = \left(-\frac{2}{3}\right)(0.1)
\]

\[
df = -\frac{2}{30}
\]
Linearization (Tangent Plane)

In one variable the linearization was:

\[ L(x) = f'(x_0)(x - x_0) + f(x_0) \]

(equation of tangent line.

Similarly for functions of 2 variables, a linearization is a "linear" surface, that is, a plane, the tangent plane. So, for \( f(x,y) \) differentiable at \( P_0(x_0, y_0) \), by definition of differentiability:

\[
\Delta f = f(x,y) - f(x_0, y_0) = f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y
\]

\[
f(x,y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0)
\]

(linearization of \( f(x,y) \) at \( (x_0, y_0) \))

This notion extends to more variables; the linearization of the differentiable function \( f(x,y,z) \) at \( P_0(x_0, y_0, z_0) \) is:

\[
\omega = \omega(x,y,z) \quad \text{Let} \quad f(x_0, y_0, z_0) = f(P_0)
\]

\[
L(x,y,z) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0)
\]
Example. Find the linearization of \( f(x,y) = x^2 - xy + \frac{1}{2} y^2 + 3 \)

at the point \((x_0, y_0) = (3, 2)\)

\[
\nabla f = \langle 2x - y, -x + y \rangle \\
\nabla f|_{(3,2)} = \langle 2(3) - 2, -3 + 2 \rangle = \langle 4, -1 \rangle
\]

\[
f(x_0, y_0) = f(3, 2) = 3^2 - 3(2) + \frac{1}{2} 2^2 + 3 = 9 - 6 + 2 + 3 = 8
\]

\[
L(x, y) = 8 + 4(x - 3) - (y - 2)
\]

**Differentials** We want to approximate the change in \( f \) from \((x_0, y_0)\) to \((x_0 + \Delta x, y_0 + \Delta y)\)

\[
\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f_x (x - x_0)^\Delta x + f_y (y - y_0)^\Delta y = \Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y}
\]

This is called the total differential of \( f \).

**Examples** Suppose a cylindrical can is designed to have radius of 1 in. and a height of 5 in, but \( r \) and \( h \) are off by 0.03 and -0.1 in. respectively.

Estimate the resulting change in the volume of the can.

\[
V = \pi r^2 h
\]

\[
\frac{dV}{dr} = 2\pi rh \quad \frac{dV}{dh} = \pi r^2
\]

\[
dV = V_r \cdot dr + V_h \cdot dh
\]

\[
dV(1.5) = V_r (1.5) \cdot 0.03 + V_h (1.5) \cdot (-0.1)
\]

where \( dr = 0.03 \) & \( dh = -0.1 \)

End of 14.6