Example. The plane $y=2$ intersects the paraboloid $z = x^2 + y^2 = f(x, y)$ in a parabola. Find the slope of the tangent line to the parabola at the point $(1,2,5)$.

Need $\frac{\partial (x^2+y^2)}{\partial x} \bigg|_{(1,2,5)}$

$m = 2x \bigg|_{(1,2,5)} = 2(1) = 2$

Slope at $(1,2,5)$ is $\frac{\partial f(1,2)}{\partial x}$

The same principle holds for more variables; to find the rate of change with respect to a variable fix the rest of the variables (consider them as constants).

\[ w = f(x, y, z) = x \sin(yz) \]

\[ f_x = \sin(yz) \quad f_{xy} = z \cos(yz) \quad f_{xyz} = -z^2 \sin(yz) \]

\[ f_y = xz \cos(yz) \quad f_{yz} = x \cos(yz) + xz(-\sin(yz))y \]

\[ f_z = xy \cos(yz) \quad f_{zx} = y \cos(yz) \]

\[ f_{yx} = z \cos(yz) \]
Second-Order Partial Derivatives

Notation:  \[
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} \quad \frac{\partial f}{\partial x} \quad \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}
\]

\[
\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \quad \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}
\]

Note the order: \[
\frac{\partial^2 f}{\partial x \partial y} \quad \frac{\partial^2 f}{\partial y \partial x}
\]

\{ \text{first w.r.t. } y, \text{second w.r.t. } x. \}

f_{yx}

Theorem If \( f(x,y) \) and its partial derivatives \( f_x, f_y, f_{xy} \) and \( f_{yx} \)
are defined throughout an open region containing a point \((a,b)\)
and are all continuous at \((a,b)\), then:

\[
f_{xy} (a,b) = f_{yx} (a,b)
\]

Example. Find \( \frac{\partial^2 \omega}{\partial x \partial y} \) if \( \omega = xy + \frac{e^y}{y^2 + 1} \)

It is easier to find \( \frac{\partial^2 \omega}{\partial y \partial x} \)

\[
= \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \left( xy + \frac{e^y}{y^2 + 1} \right) \right) = \frac{\partial}{\partial y} \left( y \right) = 1
\]
Definition. The increment/change in the function \( z = f(x,y) \)

\[
\Delta z = f(x_0 + h, y_0 + h) - f(x_0, y_0)
\]

Differentiability. A function \( z = f(x,y) \) is differentiable at \( (x,y) \)

if both \( f_x(x_0, y_0) \) and \( f_y(x_0, y_0) \) exist and \( \Delta z \) satisfies:

\[
\Delta z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y
\]

where each \( \varepsilon_1, \varepsilon_2 \to 0 \) as both \( \Delta x, \Delta y \to 0 \).

We say \( f \) is differentiable if it is differentiable at each point of its domain, we say that the graph of \( f \) is smooth.

Facts: * If \( f_x \) and \( f_y \) are continuous on an open region \( R \), then \( f \) is differentiable on \( R \).

* If \( f \) is differentiable at \( (x_0, y_0) \), then \( f \) is continuous at \( (x_0, y_0) \).
The chain rule is used for taking derivatives of composition of functions ("chain of functions").

Recall, the chain rule in one variable:

\[
\frac{d}{dt} \omega(x(t)) = \omega'(x(t)) \cdot x'(t)
\]

We also use the notation:

\[
\frac{d\omega}{dt} = \frac{d\omega}{dx} \cdot \frac{dx}{dt}
\]

Chain Rule for functions of 2 variables.
If \( w = f(x,y) \) is differentiable and if \( x = x(t), y = y(t) \) are differentiable functions of \( t \), then the composite function \( w = f(x(t), y(t)) \) is a differentiable function of \( t \) and

\[
\frac{dw}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}
\]

or

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}
\]
Proof: If \( x \) and \( y \) are differentiable, show that so is \( w = f(x, y) \) (with respect to \( t \))

We know \( f \) is differentiable (2-variable definition)

\[
\Delta w = \frac{\partial w}{\partial x} \Delta x + \frac{\partial w}{\partial y} \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y
\]

(where \( \varepsilon_1, \varepsilon_2 \to 0 \) as \( \Delta x, \Delta y \to 0 \)). Divide by \( \Delta t \)

\[
\frac{\Delta w}{\Delta t} = \frac{\partial w}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y} \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}
\]

and let

\[
\Delta t \to 0 \quad \text{then} \quad \frac{\Delta w}{\Delta t} \to \frac{dw}{dt} \quad \text{and} \quad \frac{\Delta x}{\Delta t} \to \frac{dx}{dt}, \quad \frac{\Delta y}{\Delta t} \to \frac{dy}{dt}
\]

\[
\Rightarrow \quad \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}
\]

Branch Diagram

\( \omega = f(x, y) \) (intermediate variables)

\[
\begin{align*}
\frac{dx}{dt} & \\
\frac{dy}{dt} & \\
\frac{\partial w}{\partial x} & \\
\frac{\partial w}{\partial y} & \\
\end{align*}
\]

↑ only truly independent variable
Chain Rule for functions of 3 variables

If \( w = f(x, y, z) \) is differentiable and \( x, y \) and \( z \) are differentiable functions of \( t \), then \( w = f(x(t), y(t), z(t)) \) is a differentiable function of \( t \) and:

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}
\]

Examples. Use the chain rule to find \( \frac{dw}{dt} \)

a) \( w = xy \); \( x = \cos t \) and \( y = \sin t \), \( t_o = \pi/2 \)

\[
\frac{dw}{dt} = \frac{\partial}{\partial x}(xy)x' + \frac{\partial}{\partial y}(xy)y' = y(-\sin t) + x \cos t
\]

\[
\frac{dw}{dt} = (\sin t)(-\sin t) + (\cos t)(\cos t)
\]

\[
\left. \frac{dw}{dt} \right|_{t=\pi/2} = -\sin^2(\pi/2) + \cos^2(\pi/2) = -1
\]
b) \( w = x \ y + z \); \( x = \ln (t) \), \( y = \sqrt{t} \) and \( z = \sin (3t) \); \( t_0 = 1 \).

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x}(xy+z) x' + \frac{\partial w}{\partial y}(xy+z) y' + \frac{\partial w}{\partial z}(xy+z) z'
\]

\[
\frac{dw}{dt} = y \frac{1}{\sqrt{t}} + x \frac{1}{2\sqrt{t}} + 1.3 \cos (3t)
\]

\[
= -\frac{1}{4} + \ln (1) \frac{1}{2\sqrt{1}} + 1.3 \cos (3\cdot 1)
\]

\[
\frac{dw}{dt} = 1 + 3 \cos 3
\]

Other Combinations:

\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r}
\]

\[
\frac{w}{=f(x,y)}; \ x=x(r,s), \ y=y(r,s), \ \ z=z(r,s)
\]

\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}
\]

\[
\frac{w}{=f(x,y)}; \ x=x(r,s), \ y=y(r,s), \ \ z=z(r,s)
\]

\[
\frac{dw}{dr} = \frac{d}{dx} \frac{2x}{2r}
\]

\[
\frac{dw}{ds} = \frac{d}{dx} \frac{dx}{ds}
\]
Example.

Assume \( w = f(S^3 + t^2) \) and \( f'(x) = e^x \). Find \( \frac{dw}{dt} \) and \( \frac{dw}{ds} \).

\[ X = S^3 + t^2 \]

\[ \frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt} = e^{S^3 + t^2} \cdot (2t) \]

\[ \frac{dw}{ds} = \frac{dw}{dx} \cdot \frac{dx}{ds} = e^{S^3 + t^2} \cdot (3s^2) \]

Example. Sand is pouring out forming a cone. At a given moment the rate of sand flowing is 10 ft/min., the radius is 3 ft. and the radius is changing 1 ft/min, and the height is changing 2 ft/min. What is the volume of the sand at that moment?