13.4 Curvature and Normal Vectors of a Curve.

**Goal:** Measure how quickly a curve is turning (bending).

We'll need the **unit tangent vector**, recall \( \mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \)

The curvature is a measure of how "quickly" a curve changes direction at a point on the curve, hence the following definition.

**Definition.** If \( \mathbf{T} \) is the unit tangent vector of a smooth curve, the curvature function of the curve is:

\[
K = \left| \frac{d\mathbf{T}}{ds} \right|
\]

We shall simplify this formula into a more convenient one.

\[
K = \left| \frac{d\mathbf{T}}{dt} \cdot \frac{dt}{ds} \right| = \left| \frac{d\mathbf{T}}{dt} \cdot \frac{1}{ds/dt} \right| = \left| \frac{d\mathbf{T}}{dt} \cdot \frac{1}{|\mathbf{r}'(t)|} \right| = \left| \frac{d^2\mathbf{r}}{dt^2} \cdot \frac{1}{|\mathbf{r}'(t)|} \right|
\]

\[
K = \frac{1}{|\mathbf{r}'(t)|} \left| \frac{d\mathbf{T}}{dt} \right|
\]
Example. (We show a straight line has zero curvature)
If a line L goes through a point \((c_1, c_2, c_3)\) \((\text{with position vector} \ C= \langle c_1, c_2, c_3 \rangle)\) and is parallel to \(\vec{v}\), we know its vector equation is:
\[
\vec{r}(t) = \vec{c} + t \vec{v}
\]

1st Find \(T = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}\)
\[
\vec{r}'(t) = \vec{v} \implies \vec{T} = \frac{\vec{v}}{|\vec{v}|} \implies \frac{d\vec{T}}{dt} = 0
\]
\[
|\frac{d\vec{T}}{dt}| = |\vec{0}| = 0
\]
Thus \(K = 0\)

Example. Find the curvature of a circle with radius \(a > 0\)

Parametrization of the circle: \(\vec{r}(t) = \langle a \cos t, a \sin t \rangle\)
\[
\vec{r}'(t) = \langle -a \sin t, a \cos t \rangle
\]
\[
|\vec{r}'(t)| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = a
\]
\[
\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \langle -\sin t, \cos t \rangle
\]
\[
\frac{d\vec{T}}{dt} = \langle -\cos t, -\sin t \rangle, \quad |\frac{d\vec{T}}{dt}| = 1
\]
\[
K = \frac{1}{|\vec{r}'(t)|} \cdot |\frac{d\vec{T}}{dt}| = \frac{1}{a}
\]

Curvature of a circle of radius \(a\):
\[
K = \frac{1}{a}
\]

In general, \(K\) is a function of \(t\).
Recall the following fact (we proved earlier)

When a vector function has constant length it is orthogonal to its derivative.

Note \[ |\vec{T}| = 1 \]

\[ |\vec{T}|^2 = 1 \Rightarrow \vec{T} \cdot \vec{T} = 1 \]

Recall \[ 2 \vec{T} \cdot \vec{T}' = 0 \] 

\[ \frac{d}{ds} \vec{T} \cdot \vec{T} + \vec{T} \cdot \frac{d\vec{T}}{ds} = 0 \]

\[ 2 \vec{T} \cdot \frac{d\vec{T}}{ds} = 0 \]

**Definition** At a point where \( k \neq 0 \), the principal normal unit vector for a smooth curve on a plane is:

\[ \vec{N} = \frac{1}{k} \frac{d\vec{T}}{ds} \]

We simplify:

\[ \vec{N} = \frac{1}{k} \left( \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds} \right) = \frac{1}{k} \frac{d\vec{T}}{dt} \cdot \frac{1}{ds/dt} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \cdot \frac{d\vec{T}}{dt} \cdot \frac{1}{|\vec{r}'(t)|} \]

\[ \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} \quad \text{or} \quad \vec{N} = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} \]

\( \vec{N} \) always points in the direction in which \( \vec{T} \) is turning
(For the following we assume we are working with a plane curve...)

**Circle of Curvature or Osculating Circle** at a point P (where \( k \neq 0 \)) is the circle on the plane of the curve such that:

1. Is tangent to the curve at P.
2. Has the same curvature at P as the curve.
3. Lies toward the concave side of the curve.

![Diagram showing a point P on a curve with tangent T and osculating circle with curvature \( k \).]

\[ \text{If this circle has curvature } k \]
\[ \text{(same as the curve). Radius will be: } \]
\[ \frac{1}{k} \]

**Definition.** The radius of curvature at P is the radius of the circle of curvature.

\[ \rho = \frac{1}{k} \]

**Toy Fact...**

Osculating = kissing (kissing circle)

from Latin osculus = kiss 😘
Examples
High curvature $\rightarrow$ Small $\rho$ vs. Low curvature $\rightarrow$ large $\rho$

Find the graph of the osculating circle, at the origin, of the parabola $y = x^2$ (at the origin).

$\vec{r}(t) = \langle t, t^2 \rangle$; $\vec{r}'(t) = \langle 1, 2t \rangle$; $|\vec{r}'(0)| = \sqrt{1+4t^2}$

$\Rightarrow \vec{T} = \frac{1}{\sqrt{1+4t^2}} \hat{x} + \frac{2t}{\sqrt{1+4t^2}} \hat{j}$

$\frac{d\vec{T}}{dt} = \left[ \frac{1}{2} (1+4t^2)^{-3/2} 8t \hat{x} + \left[ \frac{2\sqrt{1+4t^2} - 2t \frac{1}{2} (1+4t^2)^{1/2} e_t}{(1+4t^2)} \right] \hat{j} \right]$

at $t = 0$ $\left| \frac{d\vec{T}}{dt}(0) \right| = \left| 2\hat{j} \right| = 2$

$\vec{r}'(0) = 1 \Rightarrow k = 2$

$\rho = \frac{1}{2}$.

Note at origin $\vec{T} = \hat{x} \Rightarrow \vec{N} = \hat{j}$ (at the origin)

Equation of the arc of curvature:
Center @ $(0, \frac{1}{2})$ radius $\frac{1}{2}

x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$